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Analysis 1

CHAPTER 1

FIELD OF REAL NUMBERS

1.1 Introduction

1.1.1 Sets

Let S be a set

- If x is an element of S, then we write $x \in S$ (x belongs to S), otherwise we write $x \notin S$ (x does not belongs to S).
- A set A is called a subset of S, if each element of A is also an element of S, that is $a \in A$ then $a \in S$.
  To denote that A is a subset of S we write $A \subset S$.
  If $A \subset B$ and $B \subset A$ then $A = B$.
- Let A and B two subsets of S. The union of A and B is the set
  $$A \cup B = \{x \in S, x \in A \,\text{or}\, x \in B\}$$
  and the intersection of A and B is the set
  $$A \cap B = \{x \in S, x \in A \,\text{and}\, x \in B\}$$
- The empty set is the set that does not contain any elements, and is denoted by $\emptyset$.
  We note that $\emptyset \subset S$ for any set S.
- A and B are disjoint if $A \cap B = \emptyset$.
- The complement of A in S is the set
  $$S \backslash A = \{x \in S, x \notin A\}$$
  (S excluded A)
- The cartesian product of A and B, denoted by $A \times B$ is the set of ordered pairs (a, b) where $a \in A$ and $b \in B$ in other words
  $$A \times B = \{(a, b), a \in A \,\text{and}\, b \in B\}$$
- The power set of S is the set of all subsets of S and is denoted by $\mathcal{P}(S)$ or 2^S and we have

Example 1.1. *Let* $S = \{1, 2, 3\}$. *Then*

$$\mathcal{P}(S) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, S\}.$$

Notations

- $\mathbb{N}$ is the set of natural numbers $\{0, 1, 2, ...\}$.
- $\mathbb{Z}$ is the set of relative integers $\{..., -2, -1, 0, 1, 2, ...\}$.
- $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$ and $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$.

1.1.2 Set of rational numbers $\mathbb{Q}$

By definition, the set of rational numbers is

$$\mathbb{Q} = \left\{ \frac{p}{q}; p \in \mathbb{Z}, q \in \mathbb{N}^* \right\}.$$

Example 1.2. $0, -1, \frac{2}{5}, \frac{-3}{4}$ *are rational numbers*

Decimal numbers are rational numbers of the form $\frac{p}{10^n}; p \in \mathbb{Z}, n \in \mathbb{N}$.

Example 1.3. $0,5 = \frac{5}{10}, -1, \frac{6}{25} = \frac{24}{10^2}$ *are decimal numbers*

1.2 Mathematical induction

Lemma 1.1. *Every non-empty subset of* $\mathbb{N}$ *contains a smallest element.*

Theorem 1.1. *Let* $S \subset \mathbb{N}$ *be a set such that* $0 \in S$, *and if* $k \in S$ *then* $k + 1 \in S$. *Then* $S = \mathbb{N}$.

Mathematical induction

Let $P(n)$ be a proposition depending on $n \in \mathbb{N}$. It can, for each n, be true or false. To show that $P(n)$ is true for all n, it suffices to verify that $P(0)$ is true then verify that $P(n + 1)$ is true assuming that $P(n)$ is true.

Example 1.4. *Let* $r \in \mathbb{R} \setminus \{1\}$. *Let us show, by induction, that for all* $n \in \mathbb{N}$,

$$1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}.$$

The formula is trivial if $n = 0$. *Assuming that*

$$1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r},$$

we will have

$$1 + r + r^2 + \cdots + r^n + r^{n+1} = \frac{1 - r^{n+1}}{1 - r} + r^{n+1} = \frac{1 - r^{n+2}}{1 - r}.$$

> *Then the formula is true for $n+1$. By induction we have*
>
> $$\forall n \in \mathbb{N}, 1+r+r^2+\cdots+r^n = \frac{1-r^{n+1}}{1-r}.$$

Theorem 1.2. *Let $a, b \in \mathbb{R}$. For all $n \in \mathbb{N}$,*

$$a^{n+1} - b^{n+1} = (a-b)(a^n + a^{n-1}b + \cdots + ab^{n-1} + b^n) = (a-b)\sum_{k=0}^{n} a^{n-k}b^k.$$

Proof. We can assume that $a \neq 0$ and $a \neq b$. By dividing by a^{n+1}, we see that it is a question of demonstrating the equality

$$1 - \frac{b^{n+1}}{a^{n+1}} = \left(1 - \frac{b}{a}\right)\left(1 + \frac{b}{a} + \left(\frac{a}{b}\right)^2 + \cdots + \left(\frac{a}{b}\right)^n\right),$$

or again, by setting $r = b/a$ and dividing by $1-r$,

$$1+r+r^2+\cdots+r^n = \frac{1-r^{n+1}}{1-r}.$$

□

The following theorem is stated using numbers called coefficients of the binomial which are written themselves in terms of so-called factorial numbers: by definition,

$$n! = 1 \times 2 \times \cdots \times n,\ n \in \mathbb{N}^*, \quad \text{et} \quad 0! = 1$$

and

$$C_n^k = \frac{n!}{k!(n-k)!}.$$

Theorem 1.3. *Let $a, b \in \mathbb{R}$. For all $n \in \mathbb{N}$,*

$$(a+b)^n = \sum_{k=0}^{n} C_n^k a^{n-k} b^k.$$

1.3 Set of real numbers $\mathbb{R}$

Proposition 1.4. *A number is rational if and only if it admits periodic or finite decimal writing.*

Example 1.5. $\frac{3}{10} = 0,3$, $\frac{-5}{2} = -2.5$, $\frac{4}{3} = 1,3333...$

Remark 1.1. If a number is not rational, we say it is irrational.

Definition 1.1 (Set of real numbers). The set of real numbers $\mathbb{R}$ is the union of rational and irrational numbers.

Example 1.6. $2, -9, 4.5, \frac{4}{11}, \sqrt{2}, \pi, e$ *are real numbers*

1.3.1 $(\mathbb{R}, +, \cdot)$ is a commutative field

1. The addition $(+)$ in $\mathbb{R}$ satisfies the following properties:
   1) It is associative:$\forall a, b, c \in \mathbb{R}, (a+b)+c = a+(b+c)$.
   2) It has a neutral element 0: $\forall a \in \mathbb{R}, a+0 = 0+a = a$.
   3) Every real has an opposite:$\forall a \in \mathbb{R}, \exists b \in \mathbb{R} : a+b = b+a = 0$, the number b opposite a is denoted $-a$.
   4) It is commutative:$\forall a, b \in \mathbb{R}, a+b = b+a$.
2. Multiplication $(\cdot)$ in $\mathbb{R}$ verifies the following properties:
   1) It is associative:$\forall a, b, c \in \mathbb{R}, (a \cdot b) \cdot c = a \cdot (b \cdot c)$.
   2) It has a neutral element 1: $\forall a \in \mathbb{R}, a \cdot 1 = 1 \cdot a = a$.
   3) Every non-zero real has an inverse:$\forall a \in \mathbb{R}^*, \exists b \in \mathbb{R}^* : a \cdot b = b \cdot a = 1$, the number b inverse of a is denoted a^{-1} or $1/a$.
   4) It is commutative:$\forall a, b \in \mathbb{R}, a \cdot b = b \cdot a$.

In addition, we have:

1. Multiplication $(\cdot)$ in $\mathbb{R}$ is distributive relative to addition:
   $\forall a, b, c \in \mathbb{R}, a \cdot (b+c) = a \cdot b + a \cdot c$.
2. If $a \cdot b = 0$, then $a = 0$ or $b = 0$.

We say that $(\mathbb{R}, +, \cdot)$ is a commutative field.

1.3.2 $(\mathbb{R}, \leq)$ is totally ordered

Consider on $\mathbb{R}$ the relation $\leq$.
For all $a, b, c \in \mathbb{R}$, we have:

1. $a \leq a$. ($\leq$ is reflexive).
2. If $a \leq b$ and $b \leq a$, then $a = b$. ($\leq$ is antisymmetric)
3. If $a \leq b$ and $b \leq c$, then $a \leq c$.($\leq$ is transitive).

In addition we have: $\forall a, b \in \mathbb{R}, a \leq b$ or $b \leq a$.
We say that $(\mathbb{R}, \leq)$ is totally ordered.

Remark 1.2. The operations $(+)$ and $(\cdot)$ on $\mathbb{R}$ are compatible with the order relation $\leq$ in the following sense, for real numbers a, b, c, d :

. If $a \leq b$ and $c \leq d$, then $a + c \leq b + d$.

. If $a \leq b$ and $c \geq 0$, and $a \cdot c \leq b \cdot c$.

. If $a \leq b$ and $c \leq 0$, then $a \cdot c \geq b \cdot c$.

1.4 The absolute value

Definition 1.2. Let $x \in \mathbb{R}$. We define the absolute value of x as being the positive real number, denoted $|x|$ given by:

$$|x| = \begin{cases} x, & si \quad x \geq 0, \\ -x, & si \quad x < 0. \end{cases}$$

Theorem 1.5. *Let $x, y \in \mathbb{R}$ and $r \in \mathbb{R}_+^*$, we have:*

1) $|x| \geq 0,$
2) $|x| = 0, \quad ssi \quad x = 0,$
3) $|xy| = |x||y| \quad et \quad |-x| = |x|,$
4) $|x + y| \leq |x| + |y|,$
5) $\sqrt{x^2} = |x|,$
6) $||x| - |y|| \leq |x + y|,$
7) $|y - x| \leq r, \quad ssi \quad x - r \leq y \leq x + r.$

1.5 Intervals

Definition 1.3. An interval of $\mathbb{R}$ is a subset I of $\mathbb{R}$ satisfying the following property:

$$Let\, a, b \in I;\ \text{if} \quad a \leq x \leq b, \quad then\ x \in I.$$

Lemma 1.2. *Interval $\mathbb{R}$ is a set of one of the following forms:*

- $\mathbb{R} = (-\infty, +\infty)$
- $[a, +\infty) = \{x \in \mathbb{R}; x \geq a\}$
- $(a, +\infty) = \{x \in \mathbb{R}; x > a\}$
- $(-\infty, b] = \{x \in \mathbb{R}; x \leq b\}$
- $(-\infty, b) = \{x \in \mathbb{R}; x < b\}$
- $[a, b] = \{x \in \mathbb{R}; a \leq x \leq b\}$
- $(a, b] = \{x \in \mathbb{R}; a < x \leq b\}$
- $[a, b) = \{x \in \mathbb{R}; a \leq x < b\}$
- $(a, b) = \{x \in \mathbb{R}; a < x < b\}$
- $\{a\} = [a, a]$
- $(a, a) = \phi$

1.6 Upper bound, lower bound, least upper bound

Definition 1.4 (Greatest element or Maximum, Least element or Minimum)**.** Let A be a non-empty part of $\mathbb{R}$ and a real a. We say that a is:.

- the greatest element of A if $a \in A$ and $\forall x \in A, x \leq a$.
- the Least element of A if $b \in A$ and $\forall x \in A, x \geq b$.

If it exists, the greatest element of A is unique. We will denote it by $\max A$. Similarly, if it exists, the least element of A is unique and we will denote it by $\min A$.

Example 1.7.

1. $\max[a, b] = b, \min[a, b] = a$.
2. *The interval* $]a, b[$ *has neither a greatest element nor a least element.*
3. $\mathbb{N}$ *has a least element* 0 *but it does not have a greatest element.*

Definition 1.5 (Upper bound, lower bound)**.** Let A be a non-empty part of $\mathbb{R}$. A real M is an upper bound of A if:$\forall x \in A, x \leq M$. A real m is a lower bound of A if:$\forall x \in A, x \geq m$.

Example 1.8.

1. $\sqrt{2}$ *is an upper bound of* $]0, 1[$.
2. $-1; 0.5, 1.3, 2$ *are lower bounds of* $]3, +\infty[$, *but there is no upper bound.*

Definition 1.6 (Supremum or Least upper bound, Infimum or Greatest lower bound)**.** Let A be a non-empty part of $\mathbb{R}$.

1. The least upper bound of A is, if it exists, the smallest element of the set of upper bounds of A. It is denoted by $\sup A$.
2. The infimum of A is, if it exists, the greatest element of the set of lower bounds of A. It is denoted by $\inf A$.

Example 1.9. 1. 2 *is the least upper bound of* $]0, 2[$ *or of* $[0, 2]$.

2. 3 *is the greatest lower bound of* $[3, +\infty[$, *But there is no supremum.*

Lemma 1.3. *If a subset* A *of* $\mathbb{R}$ *has a supremum, then it is unique.*

Proof. If a_1 and a_2 are supremums of A, then a_1 is an upper bound of A, hence $a_2 \leq a_1$. Similarly $a_1 \leq a_2$. Therefore $a_1 = a_2$. □

Axiom of completeness

1. Every non-empty subset of $\mathbb{R}$ that is bounded above has a least upper bound.
2. Similarly: Every non-empty subset of $\mathbb{R}$ that is bounded below has a greatest lower bound

Characterization of the least upper bound

Theorem 1.6. *Let* A *be a non-empty subset of* $\mathbb{R}$ *that is bounded above, and let aa be a real number. The following two statements are equivalent:*

(1) $\sup A = a$

(2) $\begin{cases} \forall x \in A, x \leq a, & \text{and} \\ \forall \epsilon > 0, \exists x_\epsilon \in A : a - \epsilon < x_\epsilon \leq a. \end{cases}$

Proof.

1. ($\Longrightarrow$) Suppose a is the least upper bound (supremum) of A.By definition, a is an upper bound of A, which satisfies the first assertion of (2). Let $\epsilon > 0$, if $a - \epsilon$ were also an upper bound of A , we would have a $a \leq a - \epsilon$ which is false. Since $a - \epsilon$ is not an upper bound of A, there exists $x_\epsilon \in A$ such that $a - \epsilon < x_\epsilon$.

2. ($\Longleftarrow$) Now suppose that (2) is true and show that a is the least upper bound of A. It is clear that a is an upper bound of A. We need to show that it is the smallest among the upper bounds of A. Suppose for contradiction that this is not the case. Then there exists a real number a' that is an upper bound of A and $a' < a$. Therefore:

$$\forall x \in A, \quad x \leq a' < a.$$

Let $\epsilon = a - a' > 0$. Applying (2), we can assert that there exists an element $x \in A$ such that $a - \epsilon < x \leq a$, meaning $a' < x \leq a$. This contradicts the assumption that a' is an upper bound of A, there by proving the second implication by contradiction.

□

Corollary 1.7. *Let A be a non-empty subset of $\mathbb{R}$ that is bounded below, and let b be a real number, we have:*

$$\inf A = b \iff \begin{cases} \forall x \in A, x \geq b, \quad \textit{and} \\ \forall \epsilon > 0, \exists x_\epsilon \in A : b \leq x_\epsilon < b + \epsilon \end{cases}$$

1.7 Extended Real Line $\overline{\mathbb{R}}$

Definition 1.7. The extended real line, denoted as $\overline{\mathbb{R}}$, is obtained by adding two elements $+\infty$ and $-\infty$ to $\mathbb{R}$.

Notation: The order relation $\leq$ on $\overline{\mathbb{R}}$ is extended as follows:

$$\forall x \in \overline{\mathbb{R}}, \quad x \leq +\infty \text{ and } x \geq -\infty.$$

Remark 1.3. $\overline{\mathbb{R}}$ has a greatest element: $+\infty$, and a least element: $-\infty$.

1.8 Archimedean Property

Theorem 1.8 (Archimedean Property). *$\mathbb{R}$ satisfies the following property, known as the Archimedean property:*

$$\forall x \in \mathbb{R}_+^*, \forall y \in \mathbb{R}, \exists n \in \mathbb{N} : nx \geq y.$$

Proof. (by contradiction): Assume the negation of the statement. That is, suppose there exist $x \in \mathbb{R}_+^*$ and $y \in \mathbb{R}$ such that:

$$\forall n \in \mathbb{N} : nx < y.$$

Define the set $\mathcal{A} = \{nx \mid n \in \mathbb{N}\}$. This set is non-empty and bounded above by y. By the completeness axiom, $\mathcal{A}$ has a least upper bound $a \in \mathbb{R}$. Specifically:

$$\forall n \in \mathbb{N} : nx \leq a,$$

which implies

$$\forall n \in \mathbb{N} : (n+1)x \leq a.$$

Thus,

$$\forall n \in \mathbb{N} : nx \leq a - x.$$

Since $a - x$ is also an upper bound of $\mathcal{A}$ and $x > 0$, we have $a - x < a$. Therefore, a is not the smallest upper bound of $\mathcal{A}$, contradicting the assumption that it is the least upper bound. □

1.9 Integer Part

Lemma 1.4. *Let x a real. There exists a unique integer p such that:*

$$p \leq x < p+1.$$

This integer is called the integer part of x, denoted $E(x)$ or $[x]$.

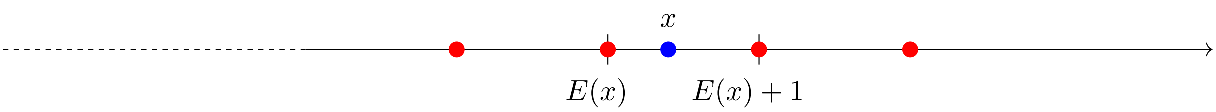

Figure 1.1: Integer Part

Example 1.10. $E(1.2) = 1, E(-0.6) = -1, E(\pi) = 3$.

Proof. Let $x \in \mathbb{R}$.

Consider the set $\mathbf{A} = \{n \in \mathbb{Z} \mid n \leq x\}$. To show that the integer part p of x exists, we need to demonstrate that the set $\mathbf{A}$ has a greatest element. We have:

1. $\mathbf{A} \neq \emptyset$: - If $x \geq 0$, then $0 \leq x$ and thus $0 \in \mathbf{A}$. - If $x < 0$, then $-x \in \mathbb{R}_+^*$. According to the Archimedean property, there exists $n \in \mathbb{N}$ such that $n \cdot 1 \geq -x$, hence $-n \leq x$. Thus, $-n \in \mathbf{A}$.

2. $\mathbf{A}$ is bounded above by x: - By the Archimedean property, there exists an integer greater than x, implying $\mathbf{A}$ is an upper-bounded subset of $\mathbb{Z}$. □

Remark 1.4. The following inequalities are often useful in exercises:

$$\forall x \in \mathbb{R}, \quad E(x) \leq x < E(x)+1 \quad \text{and} \quad x-1 < E(x) \leq x.$$

Definition 1.8. The integer part function E, or $[\cdot]$, maps each real number x to its corresponding integer p, the integer part of x.

1.10 Density of $\mathbb{Q}$ in $\mathbb{R}$

Definition 1.9. A set A in $\mathbb{R}$ is dense if:

$$\forall x \in \mathbb{R}, \forall \epsilon > 0, \exists a \in A : |a - x| \leq \epsilon.$$

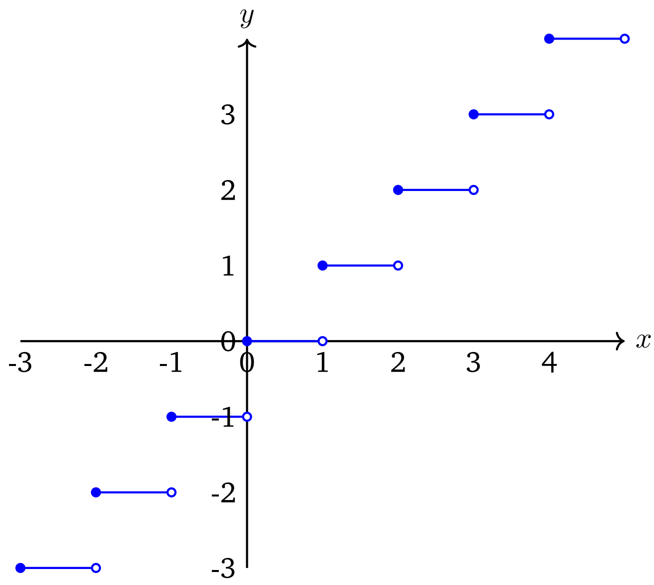

Figure 1.2: Integer Part Function

Theorem 1.9. *$\mathbb{Q}$ is dense in $\mathbb{R}$.*

Proof. Let $x \in \mathbb{R}$ and $\epsilon > 0$. Consider an integer $q > 0$ such that $1/q \leq \epsilon$.

Let $p = E(qx)$. Then $p \leq qx < p + 1$, implying $p/q \leq x < (p + 1)/q$. Let $r = p/q$; r is rational.

Since $0 \leq x - r < 1/q \leq \epsilon$, we have $|r - x| \leq \epsilon$. □