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Analysis 3

University Course Notes

For Second-Year Engineering and LMD Students

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Preface

This book is intended for second-year undergraduate students in Science and Technology (ST). It covers the main topics of Analysis 3, including definite integrals, numerical and theoretical aspects of integration, series, sequences and series of functions, and Fourier series.

The book presents these topics in a clear and organized way, with the aim of helping students understand the main concepts, follow the mathematical methods, and apply their knowledge to solve exercises and problems.

It is designed to support students in their studies and to help them develop their skills in mathematical analysis.

Chapter 1

Definite and Improper Integrals- Differential Equations

1 The Definite Integral

Let $f : [a, b] \to \mathbb{R}$ be a function. The **definite integral** $\int_a^b f(x)\,dx$ measures the net signed area between the graph of f and the x-axis over $[a, b]$. We build it in three steps: first the idea with equal subintervals, then general partitions, then Darboux sums, which give a precise criterion for integrability.

1.1 Riemann Sums: the Idea

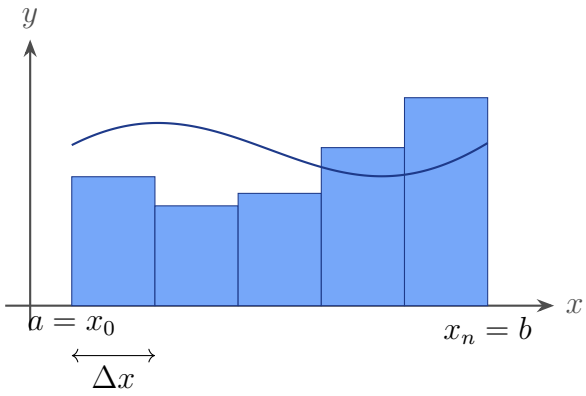
Let $f(x) \geq 0$ on $[a, b]$. We want the area A under the curve of f, above the x-axis, between the lines $x = a$ and $x = b$; we denote it $A = \displaystyle\int_a^b f(x)\,\mathrm{d}x$.

- Cut $[a, b]$ into n subintervals $[x_{i-1}, x_i]$ of equal length $\Delta x = \dfrac{b-a}{n}$, with $x_0 = a$, $x_n = b$, so that $x_i = a + i\,\Delta x$.
- Pick a point $x_i^* \in [x_{i-1}, x_i]$ in each subinterval (left end, right end, midpoint, ...).
- Form the *Riemann sum*

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x,$$

  whose i-th term is the area of the i-th rectangle: an approximation of the area.
- Take thinner and thinner rectangles:

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x.$$



1.2 Partitions and General Riemann Sums

Equal subintervals are convenient, but the definition does not require them.

Definition – Bounded Function, Partition and Mesh

A function $f : [a, b] \to \mathbb{R}$ is **bounded** if there is $M > 0$ with $|f(x)| \le M$ for all $x \in [a, b]$. A **partition** (or subdivision) of $[a, b]$ is a finite set of points

$$P = \{a = x_0 < x_1 < \cdots < x_n = b\}.$$

Put $\Delta x_i = x_i - x_{i-1}$. The **mesh** (or norm) of P is $\|P\| = \max_{1 \le i \le n} \Delta x_i$.

Definition – Riemann Sum

For a partition P and sample points $T = \{t_i\}$ with $t_i \in [x_{i-1}, x_i]$, the **Riemann sum** is

$$S(f, P, T) = \sum_{i=1}^{n} f(t_i)\, \Delta x_i.$$

For equal subintervals, $\Delta x_i = \Delta x = \frac{b-a}{n}$ and $\|P\| = \Delta x$, which is the sum of the previous subsection. The integral is the limit of $S(f, P, T)$ as $\|P\| \to 0$, when this limit exists and does not depend on the choice of P and T.

1.3 Darboux Sums and Integrability

A bounded function has a smallest and a largest value (inf and sup) on each subinterval; using them instead of arbitrary sample points traps the integral between two numbers.

Definition – Darboux Sums and Riemann Integrability

For a bounded f on $[a, b]$ and a partition P, let $m_i = \inf_{[x_{i-1}, x_i]} f$ and $M_i = \sup_{[x_{i-1}, x_i]} f$. The **lower** and **upper Darboux sums** are

$$L(f, P) = \sum_{i=1}^{n} m_i \Delta x_i, \qquad U(f, P) = \sum_{i=1}^{n} M_i \Delta x_i.$$

For every choice of sample points, $L(f, P) \le S(f, P, T) \le U(f, P)$. The function f is **Riemann integrable** on $[a, b]$ if

$$\sup_P L(f, P) = \inf_P U(f, P),$$

and this common value is $\int_a^b f(x)\, dx$. Equivalently, f is integrable if and only if for every $\varepsilon > 0$ there is a partition P with $U(f, P) - L(f, P) < \varepsilon$.

Theorem – Integrability of Continuous Functions

If f is continuous on $[a,b]$, then f is Riemann integrable on $[a,b]$. More generally, every bounded function with at most finitely many discontinuities on $[a,b]$ is Riemann integrable.

Example 1 – Darboux Sums of x^2

Let $f(x)=x^2$ on $[0,1]$ with n equal subintervals. Since f is increasing, m_i is attained at the left end and M_i at the right end:

$$L_n=\frac{1}{n}\sum_{i=1}^{n}\left(\frac{i-1}{n}\right)^2, \qquad U_n=\frac{1}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^2,$$

so

$$U_n-L_n=\frac{1}{n^3}\sum_{i=1}^{n}\left(i^2-(i-1)^2\right)=\frac{1}{n^3}\sum_{i=1}^{n}(2i-1)=\frac{1}{n}\xrightarrow[n\to\infty]{}0.$$

Hence x^2 is integrable on $[0,1]$, and the integral lies between L_n and U_n for every n.

1.4 Computing Integrals as Limits of Sums

For an integrable function, any choice of partitions with $\|P\|\to 0$ works, in particular equal subintervals with right endpoints.

Theorem – Right Riemann Sum Limit Formula

For f continuous on $[a,b]$ and $\Delta x=\frac{b-a}{n}$:

$$\int_a^b f(x)\,dx=\lim_{n\to\infty}\sum_{k=1}^{n} f\left(a+k\frac{b-a}{n}\right)\frac{b-a}{n}.$$

For $[a,b]=[0,1]$ this reads $\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right)=\int_0^1 f(x)\,\mathrm{d}x.$

Example 2 – Riemann Sum for $f(x)=x^2$

Take $f(x)=x^2$ on $[0,1]$, $t_k=k/n$:

$$S_n=\sum_{k=1}^{n}\left(\frac{k}{n}\right)^2\frac{1}{n}=\frac{1}{n^3}\sum_{k=1}^{n}k^2=\frac{1}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}\xrightarrow[n\to\infty]{}\frac{1}{3}.$$

This matches $\int_0^1 x^2\,dx=\left[\frac{x^3}{3}\right]_0^1=\frac{1}{3}$, and it lies between L_n and U_n of the previous example.

1.5 Properties of the Definite Integral

Property – Linearity

If f, g are integrable on $[a, b]$ and $\alpha, \beta \in \mathbb{R}$, then

$$\int_a^b \big(\alpha f(x) + \beta g(x)\big)\, dx = \alpha \int_a^b f(x)\, dx + \beta \int_a^b g(x)\, dx.$$

Property – Additivity over Intervals

If f is integrable on an interval containing a, b, c, then

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx.$$

Property – Monotonicity and the Triangle Inequality

If $f \leq g$ on $[a, b]$, then $\int_a^b f(x)\, dx \leq \int_a^b g(x)\, dx$. Moreover,

$$\left| \int_a^b f(x)\, dx \right| \leq \int_a^b |f(x)|\, dx.$$

Property – Orientation Convention

By convention, $\int_b^a f(x)\, dx = -\int_a^b f(x)\, dx$, and $\int_a^a f(x)\, dx = 0$.

Theorem – Mean Value Theorem for Integrals

If f is continuous on $[a, b]$, there exists $c \in [a, b]$ such that

$$\int_a^b f(x)\, dx = f(c)(b - a).$$

The quantity $\mu = \frac{1}{b-a} \int_a^b f(x)\, dx$ is called the **average value** of f on $[a, b]$.

1.6 Fundamental Theorems of Calculus

Theorem – Fundamental Theorem of Calculus (FTC)

1. If $F(x) = \int_a^x f(t)\, dt$, then $F'(x) = f(x)$.
2. If $F'(x) = f(x)$, then $\int_a^b f(x)\, dx = F(b) - F(a)$.

Proposition – Integration by Parts

If u, v are C^1 functions on $[a, b]$, then

$$\int_a^b u(x)v'(x)\,dx = \big[u(x)v(x)\big]_a^b - \int_a^b u'(x)v(x)\,dx.$$

Proposition – Change of Variables (Substitution Rule)

If $\varphi : [\alpha, \beta] \to [a, b]$ is C^1 with $\varphi(\alpha) = a$, $\varphi(\beta) = b$, and f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = \int_\alpha^\beta f(\varphi(t))\,\varphi'(t)\,dt.$$

Example 3 – Integration by Parts

Compute $\int_0^1 xe^x\,dx$. Take $u = x$, $dv = e^x dx$, so $du = dx$, $v = e^x$:

$$\int_0^1 xe^x\,dx = \big[xe^x\big]_0^1 - \int_0^1 e^x\,dx = e - (e-1) = 1.$$

Example 4 – Substitution

Compute $\int_0^1 2x\sqrt{1+x^2}\,dx$. Let $t = 1 + x^2$, $dt = 2x\,dx$; as x runs from 0 to 1, t runs from 1 to 2:

$$\int_0^1 2x\sqrt{1+x^2}\,dx = \int_1^2 \sqrt{t}\,dt = \left[\frac{2}{3}t^{3/2}\right]_1^2 = \frac{2}{3}\left(2\sqrt{2} - 1\right).$$

1.7 Symmetry and Geometric Applications

Property – Integrals of Even and Odd Functions

If f is even and integrable on $[-a, a]$, then

$$\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx.$$

If f is odd, then

$$\int_{-a}^a f(x)\,dx = 0.$$

Example 5 – Using Symmetry

Evaluate

$$I = \int_{-2}^{2} (x^5 + 3x^2 - 4x + 7)\,dx.$$

The terms x^5 and $-4x$ are odd and integrate to zero. Hence

$$I = \int_{-2}^{2} (3x^2 + 7)\,dx = 2\int_{0}^{2} (3x^2 + 7)\,dx = 2\left[x^3 + 7x\right]_0^2 = 44.$$

The important point is that symmetry can eliminate part of an integrand before any calculation.

Definition – Average Value of a Function

For an integrable function f on $[a, b]$, its **average value** is

$$f_{\text{av}} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

When f is continuous, the integral mean-value theorem guarantees a point $c \in [a, b]$ such that $f(c) = f_{\text{av}}$.

Example 6 – Area Between Two Curves

Suppose f, g are continuous on $[a, b]$ and $f(x) \geq g(x)$. The area between their graphs is

$$A = \int_a^b \big(f(x) - g(x)\big)\,dx.$$

For $f(x) = x + 2$ and $g(x) = x^2$ on $[0, 1]$,

$$A = \int_0^1 (x + 2 - x^2)\,dx = \left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_0^1 = \frac{13}{6}.$$

1.8 More Examples and Exercises on Definite Integrals

Example 7 – Integral of an Absolute Value

Evaluate $I = \int_{-2}^{2} |x - 1|\,dx$. The formula changes at $x = 1$, so we split:

$$I = \int_{-2}^{1} (1 - x)\,dx + \int_1^2 (x - 1)\,dx = \left[x - \frac{x^2}{2}\right]_{-2}^{1} + \left[\frac{x^2}{2} - x\right]_1^2 = \frac{9}{2} + \frac{1}{2} = 5.$$

A piecewise definition is necessary whenever an absolute value occurs inside an integral.

Example 8 – A Parameter-Dependent Integral

For $a > 0$,

$$I(a) = \int_0^1 e^{-ax}\,dx = \left[-\frac{1}{a}e^{-ax}\right]_0^1 = \frac{1 - e^{-a}}{a}.$$

Such integrals occur frequently in applications and lead naturally to integral transforms (Chapter 6).

2 Improper Integrals

A proper integral $\int_a^b f(x)\,dx$ needs two things: a **finite interval** and a **bounded function**. If one of them fails, the integral is **improper**. In this section we define such integrals, build a toolbox of convergence tests, and finish with exercises.

2.1 Definitions

Definition – Improper Integral

Type 1 (infinite interval). If f is continuous on $[a, +\infty)$, respectively on $(-\infty, b]$,

$$\int_a^{+\infty} f(x)\,dx = \lim_{t\to+\infty}\int_a^t f(x)\,dx, \qquad \int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx.$$

Type 2 (unbounded function). If f is continuous on $[a, b)$ and unbounded near b, respectively continuous on $(a, b]$ and unbounded near a,

$$\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx, \qquad \int_a^b f(x)\,dx = \lim_{t\to a^+}\int_t^b f(x)\,dx.$$

The integral **converges** if the limit exists and is a finite number; otherwise (infinite limit, or no limit) it **diverges**.

Two bad points. If the integral is improper at both ends, or at an interior point c, we split it at a point c and study each piece separately, with *independent* limits:

$$\int_{-\infty}^{+\infty} f = \int_{-\infty}^{c} f + \int_c^{+\infty} f.$$

The integral converges only if **both** parts converge.

Warning

1. $\int_{-1}^1 \frac{dx}{x}$ is **not** 0. The function is unbounded at 0, so we must split; both $\int_0^1 \frac{dx}{x}$ and $\int_{-1}^0 \frac{dx}{x}$ diverge, hence the integral diverges.
2. Symmetric limits are not allowed: $\int_{-R}^R x\,dx = 0$ for every R, yet $\int_{-\infty}^{+\infty} x\,dx$ diverges because $\int_0^\infty x\,dx = \infty$.

Example 1 – Four Basic Computations

1. $\displaystyle\int_1^\infty \frac{dx}{x^2} = \lim_{b\to\infty}\Big[-\frac{1}{x}\Big]_1^b = \lim_{b\to\infty}\Big(1-\frac{1}{b}\Big) = 1$. Converges.
2. $\displaystyle\int_1^\infty \frac{dx}{x} = \lim_{b\to\infty} \ln b = +\infty$. Diverges.
3. $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{t\to 0^+}\big[2\sqrt{x}\big]_t^1 = \lim_{t\to 0^+}(2-2\sqrt{t}) = 2$. Converges.
4. $\displaystyle\int_{-\infty}^\infty \frac{dx}{1+x^2} = \lim_{a\to-\infty}\big[\arctan x\big]_a^0 + \lim_{b\to\infty}\big[\arctan x\big]_0^b = \frac{\pi}{2}+\frac{\pi}{2} = \pi$. Converges.

2.2 Reference Integrals

These are the "reference integrals": we compare every new integral with them.

Theorem – p-Integrals

Integral	Converges when	Diverges when
$\displaystyle\int_1^\infty \frac{dx}{x^p}$	$p>1$ (value $\frac{1}{p-1}$)	$p\le 1$
$\displaystyle\int_0^1 \frac{dx}{x^p}$	$p<1$ (value $\frac{1}{1-p}$)	$p\ge 1$
$\displaystyle\int_a^b \frac{dx}{(x-a)^p}$ and $\displaystyle\int_a^b \frac{dx}{(b-x)^p}$	$p<1$	$p\ge 1$

Proof (first row). If $p\neq 1$, then $\int_1^b x^{-p}\,dx = \dfrac{b^{1-p}-1}{1-p}$.

- If $p>1$: $b^{1-p}\to 0$, so the limit is $\frac{1}{p-1}$: converges.
- If $p<1$: $b^{1-p}\to\infty$: diverges.
- If $p=1$: $\int_1^b \frac{dx}{x} = \ln b\to\infty$: diverges.

The other rows are proved in the same way (the third follows from the second by translation or reflection).

Memory trick. At **infinity** we want the function to go to zero **fast** (large p). Near a **bad point** we want it to blow up **slowly** (small p).

Theorem – Exponential Reference Integral

$$\int_0^\infty e^{-ax}\,dx = \frac{1}{a} \quad (a>0), \qquad \text{and it diverges if } a\le 0.$$

Indeed $\int_0^b e^{-ax}\,dx = \frac{1-e^{-ab}}{a} \to \frac{1}{a}$ when $a > 0$. Exponentials beat every power: for $a > 0$ and every n, $x^n e^{-ax} \to 0$ and $\int_0^\infty x^n e^{-ax}\,dx$ converges.

Example 2 – A Parameter Integral

For $p > 0$, $\displaystyle\int_0^1 x^{p-1}\,dx = \left[\frac{x^p}{p}\right]_0^1 = \frac{1}{p}$. This is the second row of the table with exponent $1 - p < 1$; the condition $p > 0$ is exactly the condition for convergence at 0.

2.3 Tests for Non-Negative Functions

Let $f(x) \geq 0$. Then $F(b) = \int_a^b f$ is **increasing**, so there are only two possibilities: F stays bounded (the integral converges) or $F \to +\infty$ (it diverges).

Theorem – Boundedness Test

For $f \geq 0$: $\int_a^\infty f$ converges $\iff$ $F(b) = \int_a^b f$ is bounded above.

Theorem – Comparison Test

If $0 \leq f(x) \leq g(x)$ for all $x \geq a$, then

- $\int_a^\infty g$ converges $\Rightarrow$ $\int_a^\infty f$ converges (the smaller one is below a finite number);
- $\int_a^\infty f$ diverges $\Rightarrow$ $\int_a^\infty g$ diverges (the bigger one is above an infinite number).

The same holds for Type 2 integrals, on a neighbourhood of the bad point.

Warning

The other implications give **no information**: if $\int g$ diverges, or if $\int f$ converges, we cannot conclude anything.

Example 3 – Comparison

1. $\displaystyle\int_1^\infty \frac{dx}{x^2+x+1}$: since $x^2 + x + 1 \geq x^2$, $\frac{1}{x^2+x+1} \leq \frac{1}{x^2}$, and $\int_1^\infty \frac{dx}{x^2}$ converges ($p = 2$). Converges.
2. $\displaystyle\int_2^\infty \frac{dx}{\ln x}$: for $x \geq 2$, $\ln x < x$, so $\frac{1}{\ln x} > \frac{1}{x}$, and $\int_2^\infty \frac{dx}{x}$ diverges. Diverges.
3. $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}\,(1+x)}$: $\frac{1}{\sqrt{x}(1+x)} \leq \frac{1}{\sqrt{x}}$ and $\int_0^1 x^{-1/2}dx$ converges ($p = \frac{1}{2} < 1$). Converges.

4. $\int_0^\infty e^{-x^2}\,dx$: on $[0,1]$ the integral is proper. On $[1,\infty)$, $x^2 \geq x$ gives $e^{-x^2} \leq e^{-x}$, and $\int_1^\infty e^{-x}dx$ converges. Converges.

2.4 Limit Comparison and Equivalence

Comparison needs an inequality, which can be hard to find. Limits are often easier.

Theorem – Limit Comparison Test

Let $f, g > 0$ near the critical point ($+\infty$ or the bad point), and let

$$L = \lim \frac{f(x)}{g(x)}.$$

- If $0 < L < \infty$: $\int f$ and $\int g$ have the **same nature** (both converge or both diverge).
- If $L = 0$: $\int g$ converges $\Rightarrow \int f$ converges.
- If $L = +\infty$: $\int g$ diverges $\Rightarrow \int f$ diverges.

In particular, if $f(x) \sim g(x)$ (that is $L = 1$), the two integrals have the same nature.

Useful equivalents near 0:

$$\sin x \sim x, \qquad 1 - \cos x \sim \frac{x^2}{2}, \qquad e^x - 1 \sim x, \qquad \ln(1+x) \sim x.$$

Warning

Equivalence is reliable only for functions of constant sign. For $f(x) = \frac{\sin x}{\sqrt{x}} + \frac{\sin^2 x}{x}$ and $g(x) = \frac{\sin x}{\sqrt{x}}$ we have $f \sim g$ at infinity, yet $\int_1^\infty g$ converges while $\int_1^\infty f$ diverges.

Example 4 – Limit Comparison

1. $\int_1^\infty \frac{x+1}{x^3+2}\,dx$: $f \sim \frac{x}{x^3} = \frac{1}{x^2}$, so $f/g \to 1$ with $g = \frac{1}{x^2}$. Converges.
2. $\int_1^\infty \frac{dx}{\sqrt{x^3+1}}$: $f \sim x^{-3/2}$ ($p = \frac{3}{2} > 1$). Converges.
3. $\int_1^\infty \frac{dx}{\sqrt{x^2+1}}$: with $g = \frac{1}{x}$, $f/g = \frac{x}{\sqrt{x^2+1}} \to 1$, and $\int \frac{1}{x}$ diverges. Diverges.
4. $\int_0^1 \frac{\sin x}{x^{3/2}}\,dx$: near 0, $f \sim x^{-1/2}$ ($p = \frac{1}{2} < 1$). Converges.

5. $\displaystyle\int_0^1 \frac{dx}{e^x-1}$: near 0, $e^x-1\sim x$, so $f\sim\frac{1}{x}$ $(p=1)$. Diverges.

6. $\displaystyle\int_1^\infty \frac{\ln x}{x^2}\,dx$: with $g=x^{-3/2}$, $f/g=\frac{\ln x}{\sqrt{x}}\to 0$ and $\int x^{-3/2}$ converges. Converges.

2.5 Absolute and Conditional Convergence

Now f may change sign.

Definition – Absolute and Conditional Convergence

$\int f$ converges **absolutely** if $\int |f|$ converges. It converges **conditionally** if $\int f$ converges but $\int |f|$ diverges.

Theorem – Absolute Convergence Implies Convergence

If $\int |f|$ converges, then $\int f$ converges.

Idea of proof. We have $0\le |f|+f\le 2|f|$. If $\int |f|$ converges, then $\int(|f|+f)$ converges by comparison, and $\int f=\int(|f|+f)-\int|f|$ converges.
So, to study a sign-changing function, we first test $|f|$ with the tests of Subsections 2.3 and 2.4.

Example 5 – Absolute Convergence

$\displaystyle\int_1^\infty \frac{\sin x}{x^2}\,dx$: $\left|\frac{\sin x}{x^2}\right|\le\frac{1}{x^2}$ and $\int_1^\infty\frac{1}{x^2}$ converges, so the integral converges absolutely.

Example 6 – Conditional Convergence

Study $\displaystyle\int_1^\infty \frac{\sin x}{x}\,dx$.
Step 1: it converges. Integrating by parts,

$$\int_1^b \frac{\sin x}{x}\,dx=\Big[-\frac{\cos x}{x}\Big]_1^b-\int_1^b\frac{\cos x}{x^2}\,dx.$$

The bracket tends to $\cos 1$ as $b\to\infty$, and the last integral converges absolutely ($\frac{|\cos x|}{x^2}\le\frac{1}{x^2}$).
Step 2: it is not absolutely convergent. Since $|\sin x|\ge\sin^2 x=\frac{1-\cos 2x}{2}$,

$$\frac{|\sin x|}{x}\ge\frac{1}{2x}-\frac{\cos 2x}{2x}.$$

The integral of $\frac{1}{2x}$ diverges, while that of $\frac{\cos 2x}{2x}$ converges (same method as Step 1). Hence $\int_1^\infty\frac{|\sin x|}{x}\,dx$ diverges.

Conclusion: the integral is **conditionally convergent**.

2.6 Dirichlet's Test and Abel's Test

These tests apply to integrals $\int f(x)g(x)\,dx$ in which f oscillates (like sin or cos).

Theorem – Dirichlet's Test

Suppose

1. $F(x) = \int_a^x f(t)\,dt$ is **bounded** on $[a, \infty)$, and
2. g is C^1, **monotone**, and $g(x) \to 0$ as $x \to \infty$.

Then $\displaystyle\int_a^\infty f(x)g(x)\,dx$ **converges**.

Theorem – Abel's Test

Suppose

1. $\int_a^\infty f(x)\,dx$ **converges**, and
2. g is C^1, **monotone and bounded**.

Then $\displaystyle\int_a^\infty f(x)g(x)\,dx$ **converges**.

	f	g
Dirichlet	bounded integral (like $\sin, \cos$)	monotone, tends to 0
Abel	convergent integral	monotone, bounded

Idea of proof. Dirichlet: integrate by parts, $\int_a^b fg = F(b)g(b) - \int_a^b Fg'$. The term $Fg \to 0$ (bounded times zero). Since g is monotone, $\int_a^\infty |g'| = |g(a)| < \infty$, and F is bounded, so $\int Fg'$ converges absolutely. *Abel:* a monotone bounded g has a limit ℓ; write $fg = f(g - \ell) + \ell f$, apply Dirichlet to the first term (F is bounded because $\int f$ converges) and use the hypothesis for the second.

Example 7 – Dirichlet

$\displaystyle\int_1^\infty \frac{\cos x}{\sqrt{x}}\,dx$. Take $f = \cos x$: $\int_1^x \cos t\,dt = \sin x - \sin 1$ is bounded. Take $g = x^{-1/2}$, decreasing to 0. By Dirichlet, the integral converges.

It is *not* absolutely convergent: $|\cos x| \geq \cos^2 x = \frac{1+\cos 2x}{2}$, and $\int \frac{dx}{2\sqrt{x}}$ diverges while $\int \frac{\cos 2x}{2\sqrt{x}}$ converges. So it is conditionally convergent.

Example 8 – Abel

$\int_1^\infty \frac{\sin x}{x} \arctan x \, dx$. We know that $\int_1^\infty \frac{\sin x}{x} dx$ converges (Example 2.5), and $\arctan x$ is increasing and bounded by $\frac{\pi}{2}$. By Abel, the integral converges.

Example 9 – A Family Depending on p

For which $p > 0$ does $\int_1^\infty \frac{\sin x}{x^p} \, dx$ converge?

- Dirichlet ($g = x^{-p}$ decreases to 0): it converges for **all** $p > 0$.
- Absolutely: only for $\mathbf{p > 1}$ (compare with $\frac{1}{x^p}$; for $p \leq 1$ use $|\sin x| \geq \sin^2 x$).
- Hence it is conditionally convergent for $0 < p \leq 1$.

3 Differential Equations

Definition – Differential Equation, Order and Solution

A **differential equation** (DE) is an equation involving an unknown function and one or more of its derivatives. For example,

$$y' + 2y = x, \qquad y'' - 3y' + 2y = 0.$$

The **order** of a DE is the order of the highest derivative that appears. A **solution** is a function, differentiable as many times as required, that satisfies the equation on an interval.

Definition – Linear Differential Equation

A DE is **linear of order** n if it has the form

$$a_n(x)y^{(n)} + \cdots + a_1(x)y' + a_0(x)y = g(x), \qquad a_n(x) \neq 0.$$

It is **homogeneous** if $g(x) = 0$ and **nonhomogeneous** otherwise. It has **constant coefficients** if all the a_i are constants.

Theorem – Structure of the Solutions of a Linear Equation

1. For a homogeneous linear equation, every linear combination of solutions is again a solution.
2. For a nonhomogeneous linear equation, if y_h is the general solution of the associated homogeneous equation and y_p is one particular solution, then the general solution is

$$\boxed{y = y_h + y_p}.$$

3.1 First-Order Differential Equations

3.1.1 Separable Equations

Definition – Separable Equation

A first-order equation is **separable** if it can be written as

$$y' = F(x)G(y).$$

When $G(y) \neq 0$, separate the variables and integrate:

$$\frac{dy}{G(y)} = F(x)\,dx, \qquad \int \frac{dy}{G(y)} = \int F(x)\,dx + C.$$

Warning

Dividing by $G(y)$ may lose solutions. Always check separately the constant solutions $y = y_0$ with $G(y_0) = 0$.

Example 1 – A Separable Equation

Solve $y' = xy$. For $y \neq 0$,

$$\frac{dy}{y} = x\,dx \implies \ln|y| = \frac{x^2}{2} + C \implies \boxed{y = Ce^{x^2/2}}.$$

This family also contains $y = 0$ (take $C = 0$).

Example 2 – Separable Equation with an Arctangent

Solve $y' = 2x(1 + y^2)$. Then

$$\frac{dy}{1+y^2} = 2x\,dx \implies \arctan y = x^2 + C \implies \boxed{y = \tan(x^2 + C)}$$

on any interval where the solution is defined.

3.2 Homogeneous First-Order Equations

Definition – Homogeneous First-Order Equation

A first-order equation is called **homogeneous** in this section if it can be written as

$$y' = \phi\left(\frac{y}{x}\right), \qquad x \neq 0.$$

Use the substitution $y = ux$. Then $y' = u + xu'$, so

$$u + xu' = \phi(u) \implies \frac{du}{\phi(u) - u} = \frac{dx}{x}.$$

Integrate and return to $y = ux$. Constant values of u satisfying $\phi(u) = u$ must also be checked.

Example 3 – A Homogeneous Equation

Solve $y' = 1 + y/x$, $x \neq 0$. Set $y = ux$:

$$u + xu' = 1 + u \implies xu' = 1 \implies u = \ln|x| + C.$$

Thus $\boxed{y = x(\ln|x| + C)}$.

3.3 First-Order Linear Equations

Definition – First-Order Linear Equation

The standard form is

$$y' + P(x)y = Q(x).$$

There are two useful methods of solution.

Theorem – Method 1: Integrating Factor

Define $\mu(x) = e^{\int P(x)\,dx}$. Multiplying the equation by $\mu(x)$ gives $(\mu y)' = \mu Q$. Integrating and solving for y:

$$\boxed{y = \frac{1}{\mu(x)}\left(\int \mu(x)Q(x)\,dx + C\right)}.$$

Theorem – Method 2: Homogeneous Solution plus Particular Solution

First solve $y' + P(x)y = 0$. Separating variables gives

$$\frac{dy}{y} = -P(x)\,dx \implies \boxed{y_h = Ce^{-\int P(x)\,dx}}.$$

Next, find one particular solution y_p of the full equation. One way is the **variation of the constant**: replace C by a function $C(x)$ and write $y_p = C(x)e^{-\int P(x)\,dx}$. Substituting into $y' + P(x)y = Q(x)$ gives

$$C'(x)e^{-\int P(x)\,dx} = Q(x) \implies C(x) = \int Q(x)e^{\int P(x)\,dx}\,dx,$$

so

$$\boxed{y_p = e^{-\int P(x)\,dx}\int Q(x)e^{\int P(x)\,dx}\,dx}.$$

The general solution is $\boxed{y = y_h + y_p}$, which is the same formula as in Method 1.

Example 4 – Linear Equation (Method 2)

Solve $y' + 2y = 1$. The homogeneous equation $y' + 2y = 0$ gives

$$\frac{dy}{y} = -2\,dx \implies y_h = Ce^{-2x}.$$

For a particular solution try a constant $y_p = A$. Substitution gives $2A = 1$, hence $y_p = \frac{1}{2}$. Therefore

$$\boxed{y = \frac{1}{2} + Ce^{-2x}}.$$

Example 5 – Linear Equation with an Initial Condition

Solve $y' + y = e^x$, $y(0) = 1$. By Method 1, $\mu = e^x$, so

$$(e^x y)' = e^{2x} \implies y = \frac{1}{2}e^x + Ce^{-x}.$$

The condition $y(0) = 1$ gives $C = \frac{1}{2}$. Thus

$$\boxed{y = \frac{1}{2}\left(e^x + e^{-x}\right)}.$$

3.4 Bernoulli Equations

Definition – Bernoulli Equation

A **Bernoulli equation** has the form

$$y' + P(x)y = Q(x)y^\alpha, \qquad \alpha \neq 0, 1.$$

For $y \neq 0$, divide by y^α and set $z = y^{1-\alpha}$. Since $z' = (1-\alpha)y^{-\alpha}y'$, the equation becomes the linear equation

$$\boxed{z' + (1-\alpha)P(x)z = (1-\alpha)Q(x)}.$$

Solve for z, then recover y. Check $y = 0$ separately when it satisfies the original equation.

Example 6 – A Bernoulli Equation

Solve $y' + y = xy^2$. Set $z = 1/y$. Then

$$-\frac{y'}{y^2} + \frac{1}{y} = x \implies z' - z = -x.$$

The integrating factor is e^{-x}, so $(e^{-x}z)' = -xe^{-x}$. Integration gives $z = x + 1 + Ce^{x}$, hence

$$\boxed{y = \frac{1}{x + 1 + Ce^{x}}},$$

where the denominator is nonzero. The function $y = 0$ is also a solution of the original equation.

3.5 Second-Order Linear Equations with Constant Coefficients

Definition – Second-Order Linear Equation with Constant Coefficients

Consider

$$ay'' + by' + cy = g(x), \qquad a \neq 0.$$

First solve the associated homogeneous equation $ay'' + by' + cy = 0$. Then find one particular solution y_p of the full equation. The general solution is $y = y_h + y_p$.

3.5.1 The Characteristic Equation

Theorem – Homogeneous Solutions

Trying $y = e^{rx}$ gives the **characteristic equation**

$$ar^2 + br + c = 0, \qquad \Delta = b^2 - 4ac.$$

1. If $\Delta > 0$ (two distinct real roots r_1, r_2):

$$\boxed{y_h = C_1 e^{r_1 x} + C_2 e^{r_2 x}}.$$

2. If $\Delta = 0$ (one repeated root r):

$$\boxed{y_h = (C_1 + C_2 x)e^{rx}}.$$

3. If $\Delta < 0$ (roots $r = \alpha \pm i\beta$, $\beta > 0$):

$$\boxed{y_h = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)}.$$

In all three cases $C_1, C_2 \in \mathbb{R}$.

Example 7 – Two Distinct Real Roots

Solve $y'' - y' - 2y = 0$. The characteristic equation is $r^2 - r - 2 = (r-2)(r+1) = 0$. Thus

$$\boxed{y = C_1 e^{2x} + C_2 e^{-x}}.$$

Example 8 – A Repeated Root

Solve $y'' - 4y' + 4y = 0$. The characteristic equation is $(r-2)^2 = 0$. Thus

$$\boxed{y = (C_1 + C_2 x)e^{2x}}.$$

Example 9 – Complex Roots

Solve $y'' - 2y' + 5y = 0$. The roots are $1 \pm 2i$. Thus

$$\boxed{y = e^x(C_1 \cos 2x + C_2 \sin 2x)}.$$

3.5.2 Finding a Particular Solution by Undetermined Coefficients

Proposition – Method of Undetermined Coefficients

This method is useful when $g(x)$ is a polynomial, an exponential, a sine or cosine, or a product or sum of these functions. Choose a trial function y_p of the same general type as $g(x)$, with unknown coefficients, substitute it into the equation, and compare coefficients.

Resonance rule. If a term of the trial function is already a solution of the homogeneous equation, multiply the trial function by x^m, where m is the multiplicity of the matching characteristic root ($m = 1$ for a simple root, $m = 2$ for a repeated root).

Right-hand side $g(x)$	Usual trial form for y_p
Polynomial $P_n(x)$	Polynomial of degree n
$e^{\alpha x}$	$Ae^{\alpha x}$
$e^{\alpha x}P_n(x)$	$e^{\alpha x}Q_n(x)$
$\cos \beta x$ or $\sin \beta x$	$A\cos \beta x + B\sin \beta x$
$e^{\alpha x}\cos \beta x$ or $e^{\alpha x}\sin \beta x$	$e^{\alpha x}(A\cos \beta x + B\sin \beta x)$

Here Q_n is an unknown polynomial of degree n. Apply the resonance rule when needed. For a sum of different terms, add the corresponding trial forms.

Example 10 – Polynomial Right-Hand Side

Solve $y'' - 3y' + 2y = 2x$. The homogeneous solution is $y_h = C_1e^x + C_2e^{2x}$. Since the right-hand side is linear, try $y_p = Ax + B$. Then $y_p' = A$, $y_p'' = 0$, and

$$-3A + 2(Ax + B) = 2x.$$

Comparing coefficients gives $A = 1$ and $2B - 3 = 0$, so $B = 3/2$. Therefore

$$\boxed{y = C_1e^x + C_2e^{2x} + x + \frac{3}{2}}.$$

Example 11 – Resonance

Solve $y'' - 2y' + y = e^x$. The characteristic equation is $(r-1)^2 = 0$, so $y_h = (C_1 + C_2 x)e^x$. Because e^x corresponds to the repeated root $r = 1$, try $y_p = Ax^2 e^x$. Substitution gives $A = 1/2$, so

$$\boxed{y = (C_1 + C_2 x)e^x + \frac{1}{2}x^2 e^x}.$$

3.5.3 Initial Conditions

A second-order equation usually requires two initial conditions, such as $y(x_0) = y_0$ and $y'(x_0) = y_1$. After finding the general solution, differentiate it and use both conditions to determine C_1 and C_2.

Example 12 – Initial Value Problem

Solve $y'' - 3y' + 2y = 0$ with $y(0) = 0$ and $y'(0) = 1$. The general solution is $y = C_1 e^x + C_2 e^{2x}$. The conditions give

$$C_1 + C_2 = 0, \qquad C_1 + 2C_2 = 1.$$

Thus $C_2 = 1$, $C_1 = -1$, and

$$\boxed{y = e^{2x} - e^x}.$$

3.5.4 Variation of Parameters

Theorem – Variation of Parameters

Use this method when a suitable trial function is difficult to guess. Write the equation in standard form

$$y'' + p(x)y' + q(x)y = f(x),$$

and let y_1, y_2 be linearly independent solutions of the homogeneous equation, with Wronskian

$$W(x) = y_1 y_2' - y_1' y_2 \neq 0.$$

A particular solution is

$$\boxed{y_p = -y_1 \int \frac{y_2 f}{W}\,dx + y_2 \int \frac{y_1 f}{W}\,dx},$$

and the general solution is $y = y_h + y_p$.

3.6 Euler–Cauchy Equations

Definition – Euler–Cauchy Equation

An Euler–Cauchy equation of second order has the form

$$ax^2y'' + bxy' + cy = 0.$$

On an interval where $x \neq 0$ try $y = x^r$. Since $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$, the equation becomes

$$ar(r-1) + br + c = 0.$$

Theorem – Solutions of an Euler–Cauchy Equation

Let r_1, r_2 be the roots of $ar(r-1) + br + c = 0$. For $x > 0$:

- distinct real roots: $y = C_1x^{r_1} + C_2x^{r_2}$;
- a repeated root r: $y = x^r(C_1 + C_2 \ln x)$;
- complex roots: use $x = e^t$, which transforms the equation into a constant-coefficient equation.

Example 13 – An Euler–Cauchy Equation

Solve $x^2y'' + 2xy' - 6y = 0$, $x > 0$. Try $y = x^r$:

$$r(r-1) + 2r - 6 = 0 \implies r^2 + r - 6 = (r+3)(r-2) = 0.$$

Therefore

$$\boxed{y = C_1x^{-3} + C_2x^2}.$$

4 Worked Exercises

Exercise 1. Limit of a Sequence via Riemann Sum. Compute $L = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{k}{n^2+k^2}$.

Solution. Factor out $\frac{1}{n}$:

$$L = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{\frac{k}{n}}{1 + \left(\frac{k}{n}\right)^2} = \int_0^1 \frac{x}{1+x^2}\,dx = \frac{1}{2}\ln(2).$$

Exercise 2. Integration by Parts, Twice. Compute $I = \int_0^{\pi} x^2 \sin(x)\,dx$.

Solution. Apply integration by parts twice. First with $u = x^2$, $dv = \sin(x)dx$, so $du = 2x\,dx$, $v = -\cos(x)$:

$$I = \left[-x^2\cos(x)\right]_0^{\pi} + 2\int_0^{\pi} x\cos(x)\,dx = \pi^2 + 2\int_0^{\pi} x\cos(x)\,dx.$$

Now compute $J = \int_0^\pi x\cos(x)\,dx$ with $u = x$, $dv = \cos(x)dx$, $du = dx$, $v = \sin(x)$:

$$J = [x\sin(x)]_0^\pi - \int_0^\pi \sin(x)\,dx = 0 - [-\cos(x)]_0^\pi = -2.$$

Hence $I = \pi^2 + 2(-2) = \pi^2 - 4$.

Exercise 3. Average Value. Find the average value of $f(x) = \sin(x)$ on $[0, \pi]$ and a point c where this value is attained.

Solution. The average value is

$$\mu = \frac{1}{\pi}\int_0^\pi \sin(x)\,dx = \frac{1}{\pi}\,[-\cos(x)]_0^\pi = \frac{2}{\pi}.$$

By the Mean Value Theorem for integrals, there is $c \in [0, \pi]$ with $\sin(c) = \frac{2}{\pi}$; since $\frac{2}{\pi} \approx 0.6366$, one such value is $c = \arcsin\left(\frac{2}{\pi}\right) \approx 0.690$ rad.

Exercise 4. Substitution with Trigonometric Functions. Compute $\int_0^{\pi/2} \cos^3(x)\,dx$.

Solution. Write $\cos^3(x) = \cos(x)(1 - \sin^2(x))$ and substitute $t = \sin(x)$, $dt = \cos(x)\,dx$:

$$\int_0^{\pi/2} \cos(x)(1 - \sin^2(x))\,dx = \int_0^1 (1 - t^2)\,dt = \left[t - \frac{t^3}{3}\right]_0^1 = \frac{2}{3}.$$

Property – Positivity

If f is integrable on $[a, b]$ and $f(x) \geq 0$ for every x, then

$$\int_a^b f(x)\,dx \geq 0.$$

More generally, if $m \leq f(x) \leq M$, then

$$m(b - a) \leq \int_a^b f(x)\,dx \leq M(b - a).$$

Theorem – Integral Comparison

If f and g are integrable on $[a, b]$ and $|f(x)| \leq g(x)$ for all x, where $g \geq 0$, then

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b g(x)\,dx.$$

Thus an integral can often be bounded without evaluating it exactly.

Exercise 8. Symmetry. Compute $I = \int_{-\pi}^{\pi} x^4 \cos x\,dx$.

Solution. Both x^4 and $\cos x$ are even, so $I = 2\int_0^\pi x^4 \cos x\,dx$. Integrating by parts repeatedly,

$$\int x^4 \cos x\,dx = x^4 \sin x + 4x^3 \cos x - 12x^2 \sin x - 24x \cos x + 24 \sin x.$$

Between 0 and π this gives $-4\pi^3 + 24\pi$. Therefore $I = -8\pi^3 + 48\pi$.

Exercise 9. Area Between Curves. Find the area enclosed by $y = x$ and $y = x^2$.

Solution. The curves meet where $x = x^2$, that is $x = 0$ and $x = 1$. On $[0, 1]$ we have $x \geq x^2$, so

$$A = \int_0^1 (x - x^2)\,dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{6}.$$

Exercise 10. A Parameter Integral. For $a > 0$, evaluate $I(a) = \int_0^a x(a - x)\,dx$.

Solution.

$$I(a) = \int_0^a (ax - x^2)\,dx = \left[\frac{ax^2}{2} - \frac{x^3}{3}\right]_0^a = \frac{a^3}{2} - \frac{a^3}{3} = \frac{a^3}{6}.$$

4.1 Exercises with Complete Solutions

Exercise 11. Exponential. Compute $\displaystyle\int_0^\infty e^{-3x}\,dx$.

Solution. $\int_0^b e^{-3x}\,dx = \frac{1-e^{-3b}}{3} \to \frac{1}{3}$.

Exercise 12. Logarithm at the Origin. Compute $\displaystyle\int_0^1 \ln x\,dx$.

Solution. An antiderivative of $\ln x$ is $x \ln x - x$, so $\int_t^1 \ln x\,dx = -1 - (t \ln t - t)$. As $t \to 0^+$, $t \ln t \to 0$, so the integral converges and equals -1.

Exercise 13. Logarithm over a Square Root. Determine the nature of $\displaystyle\int_0^1 \frac{\ln x}{\sqrt{x}}\,dx$ and compute it.

Solution. Near 0, $x^{3/4} \cdot \frac{|\ln x|}{\sqrt{x}} = x^{1/4}|\ln x| \to 0$, so $\frac{|\ln x|}{\sqrt{x}} = o(x^{-3/4})$. Since $\int_0^1 x^{-3/4}dx$ converges ($p = \frac{3}{4} < 1$), the integral converges (absolutely). Value: $\int_t^1 \frac{\ln x}{\sqrt{x}}dx = \left[2\sqrt{x}\ln x - 4\sqrt{x}\right]_t^1 = -4 - (2\sqrt{t}\ln t - 4\sqrt{t}) \to -4$.

Exercise 14. Comparison at Infinity. Decide: $\displaystyle\int_1^\infty \frac{x}{x^3 + 1}\,dx$.

Solution. $\frac{x}{x^3+1} \leq \frac{x}{x^3} = \frac{1}{x^2}$ and $\int_1^\infty \frac{1}{x^2}$ converges, so the integral converges by comparison.

Exercise 15. Two Bad Points. Study $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x}\,(1 + x)}$ and compute it.

Solution. Near 0, $f \sim x^{-1/2}$ (converges); at ∞, $f \sim x^{-3/2}$ (converges). So the integral converges. For its value, let $x = t^2$, $dx = 2t\,dt$:

$$\int_0^\infty \frac{2t\,dt}{t(1+t^2)} = 2\int_0^\infty \frac{dt}{1+t^2} = 2 \cdot \frac{\pi}{2} = \pi.$$

Exercise 16. Polynomial Times Exponential. Decide: $\displaystyle\int_1^\infty x^2 e^{-x}\,dx$.

Solution. Take $g = e^{-x/2}$. Then $f/g = x^2 e^{-x/2} \to 0$ and $\int e^{-x/2}$ converges, so the integral converges (Limit Comparison, $L = 0$).

Exercise 17. Bertrand Integral. For which p does $\displaystyle\int_2^\infty \frac{dx}{x(\ln x)^p}$ converge?

Solution. Let $u = \ln x$, $du = \frac{dx}{x}$. The integral becomes $\int_{\ln 2}^\infty \frac{du}{u^p}$, a p-integral at infinity: it converges if and only if $p > 1$.

Exercise 18. Singularity at an Endpoint. Decide: $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^4}}$.

Solution. The bad point is $x = 1$. Since $1-x^4 = (1-x)(1+x)(1+x^2)$ and $(1+x)(1+x^2) \to 4$ as $x \to 1$,

$$f(x) \sim \frac{1}{2\sqrt{1-x}}.$$

This is a p-integral with $p = \frac{1}{2} < 1$: it converges.

Exercise 19. Absolute Convergence. Is $\displaystyle\int_1^\infty \frac{\cos x}{x^2}\,dx$ absolutely convergent, conditionally convergent, or divergent?

Solution. $\left|\frac{\cos x}{x^2}\right| \le \frac{1}{x^2}$, which has a convergent integral. It is absolutely convergent.

Exercise 20. The Dirichlet Integral. Show that $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx$ converges.

Solution. Split at 1. On $(0, 1]$, $\frac{\sin x}{x} \to 1$ as $x \to 0$, so the integral is proper there. On $[1, \infty)$ it converges by Example 2.5. Hence the whole integral converges (its value is $\frac{\pi}{2}$).

Exercise 21. True or False. "If $\int_a^\infty f$ converges, then $f(x) \to 0$."

Solution. **False.** Take $f(x) = \sin(x^2)$ on $[1, \infty)$. With $u = x^2$ the integral becomes $\int_1^\infty \frac{\sin u}{2\sqrt{u}}\,du$, which converges by Dirichlet, but $\sin(x^2)$ does not tend to 0. (For series $a_n \to 0$ is necessary; for integrals it is not.)

4.2 Exercises on Differential Equations

4.2.1 Separable Equations

Exercise 14. Separable Equation. Solve $y' = 2xy$.

Solution. For $y \neq 0$, $\dfrac{dy}{y} = 2x\,dx$, so $\ln|y| = x^2 + C$. Hence

$$y = Ce^{x^2}, \qquad C \in \mathbb{R},$$

where $C = 0$ gives the solution $y = 0$.

Exercise 15. Logistic Equation. Solve $y' = y(1 - y)$.

Solution. The constant functions $y = 0$ and $y = 1$ are solutions. Otherwise,

$$\frac{dy}{y(1-y)} = dx \implies \int \left(\frac{1}{y} + \frac{1}{1-y}\right) dy = x + C \implies \ln\left|\frac{y}{1-y}\right| = x + C.$$

Thus $\dfrac{y}{1-y} = Ke^x$ with $K \neq 0$, that is

$$y = \frac{1}{1 + Ce^{-x}}, \qquad C \in \mathbb{R}\ (C = 0 \text{ gives } y = 1),$$

together with the solution $y = 0$.

Exercise 16. Implicit Solution. Solve $(1 + y^2)\,dy = 2x\,dx$.

Solution. Integrating both sides,

$$y + \frac{y^3}{3} = x^2 + C.$$

The solution is defined implicitly by this relation.

Exercise 17. Separable Equation with a Rational Factor. Solve $xy' = y(x^2 + 1)$, for $x \neq 0$.

Solution. For $y \neq 0$,

$$\frac{dy}{y} = \frac{x^2+1}{x}\,dx = \left(x + \frac{1}{x}\right) dx \implies \ln|y| = \frac{x^2}{2} + \ln|x| + C.$$

Hence

$$y = Cxe^{x^2/2}, \qquad C \in \mathbb{R},$$

where $C = 0$ gives $y = 0$.

4.2.2 Homogeneous and Linear First-Order Equations

Exercise 18. Homogeneous Equation. Solve $y' = \dfrac{y}{x} + 1$, for $x \neq 0$.

Solution. Set $y = ux$. Then $u + xu' = u + 1$, so $u' = \dfrac{1}{x}$ and $u = \ln|x| + C$. Hence

$$y = x\big(\ln|x| + C\big).$$

Exercise 19. Linear Equation. Solve $y' + 3y = e^x$.

Solution. The integrating factor is $\mu = e^{3x}$, so $(e^{3x}y)' = e^{4x}$. Therefore $e^{3x}y = \dfrac{e^{4x}}{4} + C$, that is

$$y = \frac{e^{x}}{4} + Ce^{-3x}.$$

Exercise 20. Linear Equation with Variable Coefficient. Solve $xy' - y = x^2$, for $x \neq 0$.

Solution. Since $\left(\dfrac{y}{x}\right)' = \dfrac{xy' - y}{x^2}$, the equation reads $\left(\dfrac{y}{x}\right)' = 1$. Hence $\dfrac{y}{x} = x + C$, that is

$$y = x^2 + Cx.$$

Exercise 21. Bernoulli Equation. Solve $y' + 2xy = xy^2$.

Solution. This is a Bernoulli equation with $\alpha = 2$. For $y \neq 0$ set $z = 1/y$; then $z' - 2xz = -x$. The integrating factor is e^{-x^2}:

$$\left(e^{-x^2}z\right)' = -xe^{-x^2} \implies e^{-x^2}z = \frac{1}{2}e^{-x^2} + C \implies z = \frac{1}{2} + Ce^{x^2}.$$

Therefore

$$y = \frac{1}{\frac{1}{2} + Ce^{x^2}} = \frac{2}{1 + 2Ce^{x^2}},$$

and $y = 0$ is also a solution.

4.2.3 Second-Order Homogeneous Equations

Exercise 22. Purely Imaginary Roots. Solve $y'' + 9y = 0$.

Solution. The characteristic equation $r^2 + 9 = 0$ gives $r = \pm 3i$. Hence

$$y = C_1 \cos 3x + C_2 \sin 3x.$$

Exercise 23. Repeated Root. Solve $y'' + 4y' + 4y = 0$.

Solution. The characteristic equation $(r + 2)^2 = 0$ has the double root $r = -2$. Hence

$$y = (C_1 + C_2 x)e^{-2x}.$$

Exercise 24. Distinct Real Roots. Solve $y'' - y' - 6y = 0$.

Solution. The characteristic equation $r^2 - r - 6 = (r - 3)(r + 2) = 0$ gives $r = 3$ and $r = -2$. Hence

$$y = C_1 e^{3x} + C_2 e^{-2x}.$$

Exercise 25. Complex Roots. Solve $y'' + 2y' + 5y = 0$.

Solution. The characteristic equation $r^2 + 2r + 5 = 0$ has $\Delta = -16 < 0$ and roots $r = -1 \pm 2i$. Hence

$$y = e^{-x}(C_1 \cos 2x + C_2 \sin 2x).$$

4.2.4 Nonhomogeneous Equations and Initial Conditions

Exercise 26. Constant Right-Hand Side. Solve $y'' - y' - 2y = 6$.

Solution. The homogeneous solution is $y_h = C_1e^{2x} + C_2e^{-x}$. Try $y_p = A$: then $-2A = 6$, so $A = -3$. Hence

$$y = C_1e^{2x} + C_2e^{-x} - 3.$$

Exercise 27. Resonance with a Repeated Root. Solve $y'' - 4y' + 4y = e^{2x}$.

Solution. The homogeneous solution is $y_h = (C_1 + C_2x)e^{2x}$. Since e^{2x} corresponds to the double root $r = 2$, try $y_p = Ax^2e^{2x}$. Substitution gives $2Ae^{2x} = e^{2x}$, so $A = \frac{1}{2}$. Hence

$$y = \left(C_1 + C_2x + \frac{x^2}{2}\right)e^{2x}.$$

Exercise 28. Trigonometric Right-Hand Side. Solve $y'' + y = \sin x$.

Solution. The homogeneous solution is $y_h = C_1\cos x + C_2\sin x$. Since $\sin x$ solves the homogeneous equation, try $y_p = x(A\cos x + B\sin x)$. Then

$$y_p'' + y_p = -2A\sin x + 2B\cos x = \sin x,$$

so $A = -\frac{1}{2}$ and $B = 0$. Hence

$$y = C_1\cos x + C_2\sin x - \frac{x}{2}\cos x.$$

Exercise 29. Initial Value Problem. Solve $y'' - 3y' + 2y = 0$ with $y(0) = 1$ and $y'(0) = 0$.

Solution. The general solution is $y = C_1e^x + C_2e^{2x}$. The conditions give $C_1 + C_2 = 1$ and $C_1 + 2C_2 = 0$, so $C_2 = -1$ and $C_1 = 2$. Hence

$$y = 2e^x - e^{2x}.$$

Exercise 30. Euler–Cauchy Equation. Solve $x^2y'' + xy' - y = 0$, $x > 0$.

Solution. Try $y = x^r$: $r(r-1) + r - 1 = r^2 - 1 = 0$, so $r = \pm 1$. Hence

$$y = C_1x + \frac{C_2}{x}.$$

1.4.3 Supplementary Exercises

Exercise 31. Riemann sums (Right Endpoint Rule)

Right endpoint rule:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\frac{b-a}{n}\sum_{k=1}^{n} f\left(a + \frac{b-a}{n}k\right).$$

For $a = 0$, $b = 1$ this reduces to

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) = \int_0^1 f(x)\,dx.$$

1) Using the Right Endpoint Rule of Riemann sum, calculate:

a) $\displaystyle\lim_{n\to+\infty}\sum_{k=1}^{n}\frac{k}{2n^2+k^2}$

Solution. Write $\dfrac{k}{2n^2+k^2}=\dfrac{1}{n}\cdot\dfrac{k/n}{2+(k/n)^2}=\dfrac{1}{n}f\!\left(\dfrac{k}{n}\right)$, with $f(x)=\dfrac{x}{2+x^2}$ on $[0,1]$. Hence

$$\sum_{k=1}^{n}\frac{k}{2n^2+k^2}=\frac{1}{n}\sum_{k=1}^{n}f\!\left(\frac{k}{n}\right)\xrightarrow[n\to\infty]{}\int_0^1\frac{x}{2+x^2}\,\mathrm{d}x=\Big[\tfrac{1}{2}\ln(2+x^2)\Big]_0^1=\frac{1}{2}\ln\frac{3}{2}.$$

b) $\displaystyle\lim_{n\to+\infty}\sum_{k=1}^{n}\frac{1}{\sqrt{n^2+kn}}$

Solution. $\dfrac{1}{\sqrt{n^2+kn}}=\dfrac{1}{n}\cdot\dfrac{1}{\sqrt{1+k/n}}=\dfrac{1}{n}f\!\left(\dfrac{k}{n}\right)$, $f(x)=\dfrac{1}{\sqrt{1+x}}$.

$$\sum_{k=1}^{n}\frac{1}{\sqrt{n^2+kn}}\xrightarrow[n\to\infty]{}\int_0^1\frac{\mathrm{d}x}{\sqrt{1+x}}=\Big[2\sqrt{1+x}\Big]_0^1=2(\sqrt{2}-1).$$

c) $\displaystyle\lim_{n\to+\infty}\sum_{k=1}^{n}\ln\left(\frac{n+k}{n}\right)^{1/n}$

Solution. Since $\ln\!\left(\frac{n+k}{n}\right)^{1/n}=\dfrac{1}{n}\ln\!\left(1+\dfrac{k}{n}\right)$, this is the Riemann sum of $f(x)=\ln(1+x)$ on $[0,1]$:

$$\sum_{k=1}^{n}\frac{1}{n}\ln\!\left(1+\frac{k}{n}\right)\xrightarrow[n\to\infty]{}\int_0^1\ln(1+x)\,\mathrm{d}x=\Big[(1+x)\ln(1+x)-(1+x)\Big]_0^1=2\ln 2-1.$$

d) $\displaystyle\lim_{n\to+\infty}\prod_{k=1}^{n}\left(1+\frac{k}{n}\right)^{1/n}$

Solution. Let $P_n=\prod_{k=1}^{n}\left(1+\frac{k}{n}\right)^{1/n}$. Then $\ln P_n=\dfrac{1}{n}\sum_{k=1}^{n}\ln\!\left(1+\dfrac{k}{n}\right)$, exactly the sum of part (c), so $\ln P_n\to 2\ln 2-1$. Therefore

$$\lim_{n\to\infty}P_n=e^{2\ln 2-1}=\frac{4}{e}.$$

Part 1 — Limits of Sums

a) $\displaystyle\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k}{2n^2+k^2}$

Factor n^2 from the denominator:

$$\frac{k}{2n^2+k^2}=\frac{1}{n}\cdot\frac{k/n}{2+(k/n)^2}.$$

This is $\sum \frac{1}{n} f(k/n)$ with $f(x) = \dfrac{x}{2+x^2}$, so

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k}{2n^2+k^2} = \int_0^1 \frac{x}{2+x^2}\,dx = \left[\frac{1}{2}\ln(2+x^2)\right]_0^1 = \frac{1}{2}\ln 3 - \frac{1}{2}\ln 2 = \boxed{\frac{1}{2}\ln\frac{3}{2}}.$$

b) $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+kn}}$

Factor n^2 inside the square root:

$$\frac{1}{\sqrt{n^2(1+k/n)}} = \frac{1}{n}\cdot\frac{1}{\sqrt{1+k/n}}.$$

With $f(x) = \dfrac{1}{\sqrt{1+x}}$:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+kn}} = \int_0^1 \frac{dx}{\sqrt{1+x}} = \left[2\sqrt{1+x}\right]_0^1 = \boxed{2\sqrt{2}-2}.$$

c) $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n}\ln\left(\frac{n+k}{n}\right)$

This is already of the form $\sum \frac{1}{n} f(k/n)$ with $f(x) = \ln(1+x)$:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n}\ln\left(\frac{n+k}{n}\right) = \int_0^1 \ln(1+x)\,dx = \Big[(1+x)\ln(1+x) - (1+x)\Big]_0^1 = (2\ln 2 - 2) - (-1) = \boxed{2\ln 2 - 1}.$$

d) $\displaystyle\lim_{n\to\infty} \left[\prod_{k=1}^{n}\left(1+\frac{k}{n}\right)\right]^{1/n}$

Taking the logarithm of the product:

$$\ln\left(\left[\prod_{k=1}^{n}\left(1+\frac{k}{n}\right)\right]^{1/n}\right) = \frac{1}{n}\sum_{k=1}^{n}\ln\left(1+\frac{k}{n}\right),$$

which is exactly the sum from part (c). Hence the logarithm of the limit equals $2\ln 2 - 1$, so

$$\lim_{n\to\infty}\left[\prod_{k=1}^{n}\left(1+\frac{k}{n}\right)\right]^{1/n} = e^{2\ln 2 - 1} = \frac{e^{\ln 4}}{e} = \boxed{\frac{4}{e}}.$$

Part 2 — Integrals via Riemann Sums

Standard sum formulas used below:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}, \qquad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}, \qquad \sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2.$$

a) $\int_0^1 (x + x^2 + x^3)\,dx$

Here $a = 0,\ b = 1$, so $x_k = \dfrac{k}{n}$ and width $= \dfrac{1}{n}$:

$$S_n = \sum_{k=1}^{n} \left(\frac{k}{n} + \frac{k^2}{n^2} + \frac{k^3}{n^3} \right) \frac{1}{n} = \underbrace{\frac{1}{n^2} \sum_{k=1}^{n} k}_{T_1} + \underbrace{\frac{1}{n^3} \sum_{k=1}^{n} k^2}_{T_2} + \underbrace{\frac{1}{n^4} \sum_{k=1}^{n} k^3}_{T_3}.$$

$$T_1 = \frac{1}{n^2} \cdot \frac{n(n+1)}{2} = \frac{n+1}{2n} \xrightarrow{n\to\infty} \frac{1}{2},$$

$$T_2 = \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6n^2} \xrightarrow{n\to\infty} \frac{1}{3},$$

$$T_3 = \frac{1}{n^4} \cdot \frac{n^2(n+1)^2}{4} = \frac{(n+1)^2}{4n^2} \xrightarrow{n\to\infty} \frac{1}{4}.$$

$$\int_0^1 (x + x^2 + x^3)\,dx = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \boxed{\frac{13}{12}}.$$

b) $\int_0^2 (x + x^2 + x^3)\,dx$

Here $a = 0,\ b = 2$, so $x_k = \dfrac{2k}{n}$ and width $= \dfrac{2}{n}$:

$$S_n = \sum_{k=1}^{n} \left(\frac{2k}{n} + \frac{4k^2}{n^2} + \frac{8k^3}{n^3} \right) \frac{2}{n} = \frac{4}{n^2} \sum_{k=1}^{n} k + \frac{8}{n^3} \sum_{k=1}^{n} k^2 + \frac{16}{n^4} \sum_{k=1}^{n} k^3.$$

$$\frac{4}{n^2} \cdot \frac{n(n+1)}{2} = \frac{2(n+1)}{n} \xrightarrow{n\to\infty} 2,$$

$$\frac{8}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} = \frac{4(n+1)(2n+1)}{3n^2} \xrightarrow{n\to\infty} \frac{8}{3},$$

$$\frac{16}{n^4} \cdot \frac{n^2(n+1)^2}{4} = \frac{4(n+1)^2}{n^2} \xrightarrow{n\to\infty} 4.$$

$$\int_0^2 (x + x^2 + x^3)\,dx = 2 + \frac{8}{3} + 4 = \boxed{\frac{26}{3}}.$$

c) $\int_0^1 e^x\,dx$

Here $x_k = \dfrac{k}{n}$, width $= \dfrac{1}{n}$:

$$S_n = \frac{1}{n} \sum_{k=1}^{n} e^{k/n}.$$

This is a geometric sum with ratio $r = e^{1/n}$:

$$\sum_{k=1}^{n} e^{k/n} = \sum_{k=1}^{n} r^k = r \cdot \frac{r^n - 1}{r - 1} = e^{1/n} \cdot \frac{e - 1}{e^{1/n} - 1},$$

since $r^n = \left(e^{1/n}\right)^n = e$. Thus

$$S_n = \frac{1}{n} \cdot e^{1/n} \cdot \frac{e-1}{e^{1/n}-1} = (e-1) \cdot e^{1/n} \cdot \frac{1/n}{e^{1/n}-1}.$$

Let $h = \dfrac{1}{n} \to 0$. Since $\dfrac{h}{e^h - 1} \to 1$ and $e^{1/n} \to 1$ as $n \to \infty$,

$$\int_0^1 e^x \, dx = \lim_{n\to\infty} S_n = (e-1) \cdot 1 \cdot 1 = \boxed{e-1}.$$

Elementary integrals.

a) $\displaystyle\int_0^1 (x + x^2 + x^3)\,\mathrm{d}x = \left[\tfrac{x^2}{2} + \tfrac{x^3}{3} + \tfrac{x^4}{4}\right]_0^1 = \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} = \dfrac{13}{12}.$

b) $\displaystyle\int_0^2 (x + x^2 + x^3)\,\mathrm{d}x = \left[\tfrac{x^2}{2} + \tfrac{x^3}{3} + \tfrac{x^4}{4}\right]_0^2 = 2 + \tfrac{8}{3} + 4 = \dfrac{26}{3}.$

c) $\displaystyle\int_0^1 e^x\,\mathrm{d}x = \left[e^x\right]_0^1 = e - 1.$

Exercise 32. Convergence of improper integrals

Determine whether the following improper integrals are convergent.

a) $\displaystyle\int_1^{+\infty} x^2\,\mathrm{d}x$

Solution. For $t > 1$, $F(t) = \displaystyle\int_1^t x^2\,\mathrm{d}x = \left[\frac{x^3}{3}\right]_1^t = \frac{t^3}{3} - \frac{1}{3}$. Since $\lim_{t\to+\infty} F(t) = +\infty$, the integral is **divergent**.

b) $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,\mathrm{d}x$

Solution. For $0 < t < 1$, $F(t) = \displaystyle\int_t^1 \frac{\mathrm{d}x}{\sqrt{x}} = \left[2\sqrt{x}\right]_t^1 = 2(1 - \sqrt{t})$. Hence $\lim_{t\to 0^+} F(t) = 2$: the integral is **convergent** and equals 2.

c) $\displaystyle\int_1^{+\infty} \ln(x)\,\mathrm{d}x$

Solution. By integration by parts, $\int \ln x\,\mathrm{d}x = x\ln x - x + C$. For $t > 1$,

$$F(t) = \int_1^t \ln x\,\mathrm{d}x = t\ln t - t + 1 = t(\ln t - 1) + 1.$$

Since $F(t) \to +\infty$ as $t \to +\infty$, the integral is **divergent**.

d) $\displaystyle\int_0^1 \frac{1}{\sqrt{1-x}}\,\mathrm{d}x$

Solution. The bad point is $x = 1$. Since $\int \frac{dx}{\sqrt{1-x}} = -2\sqrt{1-x} + C$, for $0 < t < 1$

$$F(t) = \int_0^t \frac{dx}{\sqrt{1-x}} = \Big[-2\sqrt{1-x} \Big]_0^t = 2 - 2\sqrt{1-t} \xrightarrow[t\to 1^-]{} 2.$$

The integral is **convergent** and equals 2.

e) $\displaystyle\int_{-\infty}^{+\infty} \frac{1}{x^2+1}\,dx$

Solution. The integrand is even, so $\displaystyle\int_{-\infty}^{+\infty} \frac{dx}{x^2+1} = 2\int_0^{+\infty} \frac{dx}{x^2+1}$. For $t > 0$, $\int_0^t \frac{dx}{x^2+1} = \arctan t \to \frac{\pi}{2}$ as $t \to +\infty$. Hence the integral is **convergent** and equals π.

f) $\displaystyle\int_0^{+\infty} \frac{e^{\arctan x}}{x^2+1}\,dx$

Solution. Put $u = \arctan x$, $du = \frac{dx}{x^2+1}$; for $t > 0$,

$$\int_0^t \frac{e^{\arctan x}}{x^2+1}\,dx = \int_0^{\arctan t} e^u\,du = e^{\arctan t} - 1 \xrightarrow[t\to+\infty]{} e^{\pi/2} - 1.$$

The integral is **convergent** and equals $e^{\pi/2} - 1$.

Exercise 33. Nature of improper integrals

1) Determine the nature of the following improper integrals.

a) $\displaystyle\int_0^{+\infty} \frac{1}{x^2}\,dx$

Solution. Split at $x = 1$: $\displaystyle\int_0^{+\infty} \frac{dx}{x^2} = \int_0^1 \frac{dx}{x^2} + \int_1^{+\infty} \frac{dx}{x^2}$. The first integral diverges ($p = 2 \geq 1$ at 0), while the second converges. Therefore the integral is **divergent**.

b) $\displaystyle\int_1^{+\infty} \frac{1-\cos x}{x^2}\,dx$

Solution. For $x \geq 1$, $0 \leq 1 - \cos x \leq 2$, so $0 \leq \dfrac{1-\cos x}{x^2} \leq \dfrac{2}{x^2}$. Since $\int_1^{+\infty} 2x^{-2}\,dx$ converges, the comparison test shows that the integral is **convergent**.

c) $\displaystyle\int_0^{+\infty} \frac{x^2}{x^{17/5}+1}\,dx$

Solution. *Near* $+\infty$: $\dfrac{x^2}{x^{17/5}+1} \sim \dfrac{x^2}{x^{17/5}} = \dfrac{1}{x^{7/5}}$, and $\int_1^{+\infty} \frac{dx}{x^{7/5}}$ converges since $\frac{7}{5} > 1$.
Near 0*:* the integrand is continuous (it behaves like x^2), so there is no problem there.
Hence the integral is **convergent**.

d) $\displaystyle\int_0^{+\infty} \sin\left(\frac{1}{x}\right) dx$

Solution. Split at $x = 1$.
On $(0, 1]$*:* put $u = 1/x$, $\mathrm{d}x = -\,\mathrm{d}u/u^2$:

$$\int_0^1 \sin\left(\frac{1}{x}\right)\mathrm{d}x = \int_1^{+\infty} \frac{\sin u}{u^2}\,\mathrm{d}u, \qquad \left|\frac{\sin u}{u^2}\right| \le \frac{1}{u^2},$$

so this integral converges absolutely.
On $[1, +\infty)$*:* as $x \to +\infty$, $\sin\left(\frac{1}{x}\right) \sim \frac{1}{x} > 0$, and $\int_1^{+\infty} \frac{\mathrm{d}x}{x}$ diverges, so this integral diverges by the equivalence test.
Therefore the integral is **divergent**.

e) $\displaystyle\int_0^{+\infty} \frac{\sin x}{x^{3/2}}\,\mathrm{d}x$

Solution. Split at $x = 1$.
On $(0, 1]$*:* $\sin x \sim x$ near 0, so $\dfrac{\sin x}{x^{3/2}} \sim \dfrac{1}{x^{1/2}}$, and $\int_0^1 x^{-1/2}\,\mathrm{d}x$ converges ($p = \frac{1}{2} < 1$).
On $[1, +\infty)$*:* $\left|\dfrac{\sin x}{x^{3/2}}\right| \le \dfrac{1}{x^{3/2}}$, and $\int_1^{+\infty} x^{-3/2}\,\mathrm{d}x$ converges ($p = \frac{3}{2} > 1$).
Hence the integral is **convergent** (absolutely at $+\infty$).

2) Discuss, according to the values of α and β, the nature of $\displaystyle\int_0^{+\infty} \frac{1}{x^\alpha + x^\beta}\,\mathrm{d}x$.

Solution. The integrand is symmetric in α, β, so put $m = \min(\alpha, \beta)$ and $M = \max(\alpha, \beta)$. The integrand is positive and continuous on $(0, +\infty)$, so we study the two ends separately.
Near 0*:* the smallest exponent dominates, $x^\alpha + x^\beta \sim x^m$, so $f(x) \sim x^{-m}$. By the p-integral at 0, $\int_0^1 f$ converges if and only if $m < 1$.
Near $+\infty$*:* the largest exponent dominates, $x^\alpha + x^\beta \sim x^M$ (if $m = M$ the sum is $2x^M$, still of the same order), so $f(x) \sim x^{-M}$. By the p-integral at infinity, $\int_1^{+\infty} f$ converges if and only if $M > 1$.
Hence the integral converges if and only if

$$\min(\alpha, \beta) < 1 < \max(\alpha, \beta),$$

that is $\alpha < 1 < \beta$ or $\beta < 1 < \alpha$. In all other cases it diverges (in particular if $\alpha = \beta$).

3) Study the absolute convergence of:

a) $\displaystyle\int_1^{+\infty} \frac{\cos x}{x^2}\,\mathrm{d}x$

Solution. $\left|\dfrac{\cos x}{x^2}\right| \le \dfrac{1}{x^2}$ and $\int_1^{+\infty} \frac{\mathrm{d}x}{x^2}$ converges. The integral is **absolutely convergent**.

b) $\displaystyle\int_0^1 \frac{\cos x}{\sqrt{x}}\,\mathrm{d}x$

Solution. $\left|\dfrac{\cos x}{\sqrt{x}}\right| \le \dfrac{1}{\sqrt{x}}$ on $(0, 1]$ and $\int_0^1 x^{-1/2}\,\mathrm{d}x$ converges. The integral is **absolutely convergent**.

c) $\displaystyle\int_1^{+\infty} \frac{\sin x}{x}\,\mathrm{d}x$

Solution. *Convergence.* Integrating by parts, $\int_1^b \frac{\sin x}{x}\,\mathrm{d}x = \Big[-\frac{\cos x}{x}\Big]_1^b - \int_1^b \frac{\cos x}{x^2}\,\mathrm{d}x$. The bracket tends to $\cos 1$, and the last integral converges absolutely by part a).
No absolute convergence. Since $|\sin x| \geq \sin^2 x = \frac{1-\cos 2x}{2}$,

$$\frac{|\sin x|}{x} \geq \frac{1}{2x} - \frac{\cos 2x}{2x}.$$

The integral of $\frac{1}{2x}$ diverges, while that of $\frac{\cos 2x}{2x}$ converges (same integration by parts). Hence $\int_1^{+\infty} \frac{|\sin x|}{x}\,\mathrm{d}x$ diverges.
The integral is **conditionally convergent**.

Part B: Differential Equations

Exercise 34. First-order differential equations.

Integrate the following differential equations.

1) Separable equations.

a) $(x^2+1)y' - y = 0$

Solution. The function $y \equiv 0$ is a solution. For $y \neq 0$, $\dfrac{y'}{y} = \dfrac{1}{x^2+1}$, so

$$\int \frac{\mathrm{d}y}{y} = \int \frac{\mathrm{d}x}{1+x^2} \implies \ln|y| = \arctan x + C.$$

Therefore $y(x) = Ce^{\arctan x}$, $C \in \mathbb{R}$ ($C = 0$ gives $y \equiv 0$).

b) $xy' - y\ln x = 0$

Solution. For $x > 0$ and $y \neq 0$, $\dfrac{y'}{y} = \dfrac{\ln x}{x}$, so $\ln|y| = \displaystyle\int \frac{\ln x}{x}\,\mathrm{d}x = \frac{(\ln x)^2}{2} + C$. Hence

$$y(x) = C\exp\Big(\frac{(\ln x)^2}{2}\Big), \qquad x > 0,\ C \in \mathbb{R}.$$

2) Homogeneous equations. A first-order equation $y' = f(x,y)$ with $f(\lambda x, \lambda y) = f(x,y)$ is solved by $u = \dfrac{y}{x}$, $y = ux$, $y' = u + xu'$, which gives a separable equation in u and x.

a) $y' = \dfrac{y+x}{x-y}$

Solution. With $y = ux$: $u + xu' = \dfrac{u+1}{1-u}$, so $xu' = \dfrac{u+1}{1-u} - u = \dfrac{1+u^2}{1-u}$. Separating,

$$\frac{1-u}{1+u^2}\,\mathrm{d}u = \frac{\mathrm{d}x}{x} \implies \arctan u - \frac{1}{2}\ln(1+u^2) = \ln|x| + C.$$

Since $\ln|x| + \frac{1}{2}\ln\left(1+\frac{y^2}{x^2}\right) = \frac{1}{2}\ln(x^2+y^2)$, this becomes

$$\arctan\frac{y}{x} = \frac{1}{2}\ln(x^2+y^2) + C, \qquad \text{i.e.} \qquad \sqrt{x^2+y^2} = Ce^{\arctan(y/x)}.$$

b) $2xyy' = x^2 + 2y^2$

Solution. For $x, y \neq 0$, $y' = \dfrac{x}{2y} + \dfrac{y}{x}$. With $y = ux$: $u + xu' = \dfrac{1}{2u} + u$, so $xu' = \dfrac{1}{2u}$. Separating,

$$2u\,du = \frac{dx}{x} \implies u^2 = \ln|x| + C.$$

Hence $y^2 = x^2\big(\ln|x| + C\big)$. (Check: $2yy' = 2x(\ln|x| + C) + x$, and multiplying by x gives $2xyy' = 2y^2 + x^2$.)

3) First-order linear equations.

a) $xy' + y = \cos x$

Solution. Since $(xy)' = xy' + y$, the equation reads $(xy)' = \cos x$. Hence $xy = \sin x + C$, that is

$$y = \frac{\sin x + C}{x}, \qquad x \neq 0.$$

b) $y' - 6y = x^2 + 1$

Solution. *Homogeneous solution:* $y_h = Ce^{6x}$. *Particular solution:* try $y_p = ax^2 + bx + c$. Then $y_p' - 6y_p = -6ax^2 + (2a - 6b)x + (b - 6c)$, and comparing with $x^2 + 1$:

$$-6a = 1, \quad 2a - 6b = 0, \quad b - 6c = 1 \implies a = -\frac{1}{6},\ b = -\frac{1}{18},\ c = -\frac{19}{108}.$$

Therefore

$$y = Ce^{6x} - \frac{x^2}{6} - \frac{x}{18} - \frac{19}{108}.$$

c) $y' + xy = e^{-x^2/2}$

Solution. The integrating factor is $\mu = e^{x^2/2}$, so $(\mu y)' = \mu e^{-x^2/2} = 1$. Hence $\mu y = x + C$, that is

$$y = (x + C)\,e^{-x^2/2}.$$

d) $y' - y\sin x = \sin 2x$

Solution. The integrating factor is $\mu = e^{\int(-\sin x)\,dx} = e^{\cos x}$, so

$$\left(e^{\cos x}y\right)' = e^{\cos x}\sin 2x = 2\sin x\cos x\, e^{\cos x}.$$

With $u = \cos x$, $du = -\sin x\,dx$: $\int 2\sin x\cos x\, e^{\cos x}\,dx = -2\int ue^u\,du = -2(u-1)e^u = 2(1-\cos x)e^{\cos x}$. Hence $e^{\cos x}y = 2(1-\cos x)e^{\cos x} + C$, that is

$$y = 2(1 - \cos x) + Ce^{-\cos x}.$$

Exercise 35. Second-order linear equations with constant coefficients.

Integrate the following differential equations.

a) $y'' - y' - 2y = -x^2 - 3x + e^{-x}$

Solution. *Homogeneous solution.* $r^2 - r - 2 = (r-2)(r+1) = 0$, so $y_h = C_1 e^{2x} + C_2 e^{-x}$.
Particular solution for $-x^2 - 3x$. Try $y_{p1} = ax^2 + bx + c$. Then $y''_{p1} - y'_{p1} - 2y_{p1} = -2ax^2 - 2(a+b)x + (2a - b - 2c)$, and comparing with $-x^2 - 3x$ gives $a = \frac{1}{2}$, $b = 1$, $c = 0$. So $y_{p1} = \frac{x^2}{2} + x$.
Particular solution for e^{-x}. Since -1 is a simple root of the characteristic equation, try $y_{p2} = axe^{-x}$. Then $y'_{p2} = a(1-x)e^{-x}$, $y''_{p2} = a(x-2)e^{-x}$ and $y''_{p2} - y'_{p2} - 2y_{p2} = -3ae^{-x}$. Thus $-3a = 1$ and $y_{p2} = -\frac{x}{3}e^{-x}$.
Therefore

$$y = C_1 e^{2x} + C_2 e^{-x} + \frac{x^2}{2} + x - \frac{x}{3}e^{-x}.$$

b) $y'' - 5y' + 6y = e^{2x} + \cos x$

Solution. *Homogeneous solution.* $r^2 - 5r + 6 = (r-2)(r-3) = 0$, so $y_h = C_1 e^{2x} + C_2 e^{3x}$.
Particular solution for $\cos x$. Try $y_{p1} = a\sin x + b\cos x$. Substitution gives $(5a+5b)\sin x + (5b - 5a)\cos x = \cos x$, so $a = -\frac{1}{10}$, $b = \frac{1}{10}$.
Particular solution for e^{2x}. Since 2 is a simple root, try $y_{p2} = cxe^{2x}$. Then $y'_{p2} = c(1 + 2x)e^{2x}$, $y''_{p2} = c(4+4x)e^{2x}$, and

$$y''_{p2} - 5y'_{p2} + 6y_{p2} = c\,e^{2x}\big[(4+4x) - 5(1+2x) + 6x\big] = -c\,e^{2x}.$$

Thus $c = -1$.
Therefore

$$y = C_1 e^{2x} + C_2 e^{3x} - xe^{2x} + \frac{\cos x - \sin x}{10}.$$

c) $y'' - 6y' + 9y = 3xe^{3x}$

Solution. The characteristic equation $(r-3)^2 = 0$ has the double root 3, so $y_h = (C_1 + C_2 x)e^{3x}$. Put $y = e^{3x}z$. Then $y' = e^{3x}(z' + 3z)$, $y'' = e^{3x}(z'' + 6z' + 9z)$, and the equation reduces to $z'' = 3x$. Taking $z = \frac{x^3}{2}$ gives the particular solution $y_p = \frac{x^3}{2}e^{3x}$. Hence

$$y = \Big(C_1 + C_2 x + \frac{x^3}{2}\Big)e^{3x}.$$

d) $y'' - 4y' + 5y = x + 1$

Solution. The characteristic equation $r^2 - 4r + 5 = 0$ has roots $2 \pm i$, so $y_h = e^{2x}(C_1\cos x + C_2 \sin x)$. Try $y_p = ax + b$: $-4a + 5ax + 5b = x + 1$ gives $a = \frac{1}{5}$ and $5b = 1 + \frac{4}{5}$, that is $b = \frac{9}{25}$. Hence

$$y = e^{2x}\big(C_1\cos x + C_2 \sin x\big) + \frac{x}{5} + \frac{9}{25}.$$

1.4.4 Exercises from Previous Mathematics 3 Examinations

Exercise 36. (2016). For $\alpha \in \mathbb{R}$, study the convergence of

$$I_\alpha = \int_1^{+\infty} \frac{\sin(1/x)}{x^\alpha}\,dx.$$

Solution. For $x \to +\infty$, we have $1/x \to 0$ and $\sin(1/x) \sim 1/x$. Hence

$$\frac{\sin(1/x)}{x^\alpha} \sim \frac{1}{x^{\alpha+1}}.$$

The integrand is nonnegative on $[1, +\infty)$, so the limit-comparison test for improper integrals applies. The standard integral $\int_1^{+\infty} x^{-p}\,dx$ converges exactly when $p > 1$. Here $p = \alpha + 1$, so I_α converges if and only if $\alpha + 1 > 1$, that is, if and only if $\boxed{\alpha > 0}$. It diverges for $\alpha \leq 0$.

Exercise 37. (2025).

1. Using the definition of the definite integral as a limit of Riemann sums, compute $\int_0^2 (x+1)\,dx$.

2. Using polar coordinates, compute $\iint_D e^{-(x^2+y^2)}\,dx\,dy$, where $D = \{(x, y) : x \geq 0,\ y \geq 0\}$.

Solution. **1.** Divide $[0, 2]$ into n equal intervals, $\Delta x = 2/n$, with right endpoints $x_k = 2k/n$. Then

$$\int_0^2 (x+1)\,dx = \lim_{n\to\infty} \frac{2}{n} \sum_{k=1}^{n} \left(\frac{2k}{n} + 1\right) = \lim_{n\to\infty} \left(\frac{2(n+1)}{n} + 2\right) = 4.$$

2. In polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$, and $dx\,dy = r\,dr\,d\theta$. The first quadrant is described by $0 \leq \theta \leq \pi/2$ and $0 \leq r < \infty$. Thus

$$\iint_D e^{-(x^2+y^2)} dx dy = \int_0^{\pi/2} \int_0^{\infty} e^{-r^2} r\,dr\,d\theta.$$

The radial integral is improper because its upper limit is infinite. To evaluate it, substitute $u = r^2$, so $du = 2r\,dr$:

$$\int_0^{\infty} e^{-r^2} r\,dr = \frac{1}{2}\int_0^{\infty} e^{-u} du = \frac{1}{2} \lim_{B\to\infty} [-e^{-u}]_0^B = \frac{1}{2}.$$

The limit is finite, so this improper integral converges. The angular integral is $\int_0^{\pi/2} d\theta = \pi/2$, and consequently the double integral equals $\frac{\pi}{2} \cdot \frac{1}{2} = \boxed{\pi/4}$.

Exercise 38. (2026). Using the definition of the definite integral as a limit of Riemann sums, compute $\int_0^2 (x+1)\,dx$.

Solution. With $x_k = 2k/n$ and $\Delta x = 2/n$,

$$\int_0^2 (x+1)dx = \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{2k}{n} + 1\right) \frac{2}{n} = \lim_{n\to\infty} \left(\frac{2(n+1)}{n} + 2\right) = 4.$$

Exercise 39. (2026). Compute exactly $I = \displaystyle\int_1^{\infty} \frac{dx}{x^2+1}$.

Solution. Step 1: Identify why the integral is improper. The upper endpoint is $+\infty$, so we must interpret the integral as a limit. We cannot simply substitute $+\infty$ into an antiderivative.

Step 2: Apply the definition. For $b > 1$,

$$I_b = \int_1^b \frac{dx}{1+x^2}.$$

Since $\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$, the Fundamental Theorem of Calculus gives

$$I_b = [\arctan x]_1^b = \arctan b - \arctan 1.$$

Step 3: Take the limit. As $b \to +\infty$, $\arctan b \to \pi/2$, while $\arctan 1 = \pi/4$. Therefore

$$I = \lim_{b\to+\infty} I_b = \frac{\pi}{2} - \frac{\pi}{4} = \boxed{\frac{\pi}{4}}.$$

Conclusion and rule. The defining limit exists and is finite, so the improper integral converges and has value $\pi/4$. This is the definition test for an improper integral; no comparison test is needed because an exact antiderivative is available.

Exercise 40. (2026). Study the convergence of $\displaystyle\int_1^{\infty} \frac{\sin^2 x}{x^4}\,dx$.

Solution. Step 1: Check the sign and find a simple upper bound. Since $0 \le \sin^2 x \le 1$ for every real x, and $x^4 > 0$ for $x \ge 1$,

$$0 \le \frac{\sin^2 x}{x^4} \le \frac{1}{x^4}, \qquad x \ge 1.$$

Step 2: State the comparison rule. If $0 \le f(x) \le g(x)$ for all sufficiently large x, and $\int_1^{\infty} g(x)\,dx$ converges, then $\int_1^{\infty} f(x)\,dx$ also converges. Here $f(x) = \sin^2 x/x^4$ and $g(x) = 1/x^4$ satisfy these hypotheses.

Step 3: Test the comparison integral. This is a p-integral:

$$\int_1^{\infty} \frac{dx}{x^p},$$

which converges exactly when $p > 1$. In our case $p = 4 > 1$, and explicitly

$$\int_1^{\infty} \frac{dx}{x^4} = \lim_{b\to\infty} \left[-\frac{1}{3x^3}\right]_1^b = \frac{1}{3}.$$

Conclusion. The comparison integral is finite and the given non-negative integrand is

no larger than it. By the comparison test,

$$\boxed{\int_1^\infty \frac{\sin^2 x}{x^4}\,dx \text{ converges.}}$$

The comparison proves convergence, but it does not by itself give the exact value of the integral.

Supplementary Exercise. (2024). Using a Riemann sum, calculate $L = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{k}{n^2+kn}$.

Solution. The right-endpoint Riemann-sum formula is

$$\lim_{n\to\infty} \frac{b-a}{n} \sum_{k=1}^{n} f\left(a + \frac{k(b-a)}{n}\right) = \int_a^b f(x)\,dx.$$

Rewrite the sum as $\frac{1}{n}\sum_{k=1}^{n} \frac{k/n}{1+k/n}$. This is the Riemann sum for $f(x) = x/(1+x)$ on $[0,1]$. Hence

$$L = \int_0^1 \frac{x}{1+x}\,dx = \int_0^1 \left(1 - \frac{1}{1+x}\right) dx = [x - \ln(1+x)]_0^1 = \boxed{1 - \ln 2}.$$

Supplementary Exercise. (2024). Using a Riemann sum, calculate

$$L = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{k}{n^2+kn}.$$

Solution. Rewrite each term as

$$\frac{k}{n^2+kn} = \frac{1}{n}\frac{k/n}{1+k/n}.$$

Thus the sum is the right-endpoint Riemann sum for the continuous function $f(x) = x/(1+x)$ on $[0,1]$. Therefore

$$L = \int_0^1 \frac{x}{1+x}\,dx = \int_0^1 \left(1 - \frac{1}{1+x}\right) dx = [x - \ln(1+x)]_0^1 = \boxed{1 - \ln 2}.$$

Exercise 41. (2016). Study the convergence of

$$I = \int_2^\infty \frac{dx}{x(\ln x)^2},$$

and deduce the nature of the series

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}.$$

Solution. For $b > 2$, integrate on the finite interval first:

$$\int_2^b \frac{dx}{x(\ln x)^2} = \left[-\frac{1}{\ln x}\right]_2^b = \frac{1}{\ln 2} - \frac{1}{\ln b}.$$

Taking $b \to \infty$ gives $I = 1/\ln 2 < \infty$. Hence the improper integral converges. The function $f(x) = 1/[x(\ln x)^2]$ is positive, continuous, and decreasing for $x \geq 2$; by the integral test, the series has the same nature as the integral. Therefore the series converges.

Exercise 42. (2017). For $\alpha > 0$, study the convergence and absolute convergence of

$$I_\alpha = \int_1^\infty \frac{\sin(5x)}{x^\alpha}\, dx.$$

Solution. If $\alpha > 1$, then

$$\left|\frac{\sin(5x)}{x^\alpha}\right| \leq \frac{1}{x^\alpha}, \qquad \int_1^\infty x^{-\alpha}\, dx < \infty.$$

Comparison proves absolute convergence. For $0 < \alpha \leq 1$, the Dirichlet test for improper integrals applies: $x^{-\alpha}$ is positive, decreases to zero, and the partial integrals of $\sin(5x)$ are bounded because

$$\left|\int_a^b \sin(5x)\, dx\right| = \frac{|\cos(5a) - \cos(5b)|}{5} \leq \frac{2}{5}.$$

Thus I_α converges. It is not absolutely convergent in this range: on intervals where $|\sin(5x)|$ stays bounded away from zero, the integral of $|\sin(5x)|/x^\alpha$ has the same divergence threshold as $\int_1^\infty x^{-\alpha}dx$, which diverges for $\alpha \leq 1$. Hence convergence is conditional for $0 < \alpha \leq 1$ and absolute for $\alpha > 1$.

Chapter 2

Double and Triple Integrals

2.1 Double Integrals

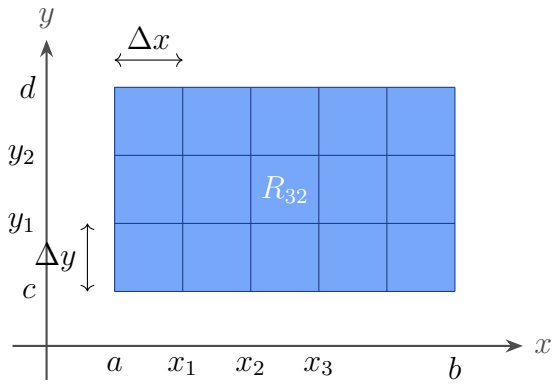
The double integral extends the definite integral to functions of two variables. It computes volumes, areas, average values and, later, masses and moments. This section builds it step by step: rectangles, iterated integrals (Fubini), general regions, change of order, polar coordinates, and general changes of variables.

2.1.1 The Double Integral over a Rectangle

Let $f(x,y)$ be defined on the rectangle $R = \{(x,y) \in \mathbb{R}^2 \mid a \le x \le b,\ c \le y \le d\} = [a,b] \times [c,d]$, and assume first that $f(x,y) \ge 0$ on R. We want the **volume** of the solid under the graph of f,

$$S = \{(x,y,z) \in \mathbb{R}^3 \mid (x,y) \in R,\ 0 \le z \le f(x,y)\}.$$

We proceed exactly as for the definite integral: cut R into small rectangles, approximate the volume over each one, add, and let the pieces shrink.



- Divide $[a,b]$ into m subintervals $[x_{i-1}, x_i]$ of length $\Delta x = \frac{b-a}{m}$, and $[c,d]$ into n subintervals $[y_{j-1}, y_j]$ of length $\Delta y = \frac{d-c}{n}$.
- R is thus cut into $m \cdot n$ small rectangles $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$, each of area $\Delta A = \Delta x\, \Delta y$.
- Pick a sample point $(x_i^*, y_j^*) \in R_{ij}$. The product $f(x_i^*, y_j^*)\Delta A$ is the volume of a box with base R_{ij} and height $f(x_i^*, y_j^*)$.

Definition – Double Integral over a Rectangle

The **double integral** of f over R is

$$\iint_R f(x,y)\,\mathrm{d}A = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*)\,\Delta A,$$

when the limit exists and does not depend on the choice of the points (x_i^*, y_j^*). If $f \ge 0$, it is the volume of S. Every continuous function on R is integrable.

Remark

The sample points are arbitrary. Common choices are the **midpoint** $\left(\frac{x_{i-1}+x_i}{2}, \frac{y_{j-1}+y_j}{2}\right)$ and a **corner** of R_{ij}, for instance (x_i, y_j).

Property – Linearity and Monotonicity

For integrable f, g on R and $\alpha, \beta \in \mathbb{R}$:

(1) $\displaystyle\iint_R (\alpha f + \beta g)\,\mathrm{d}A = \alpha \iint_R f\,\mathrm{d}A + \beta \iint_R g\,\mathrm{d}A$;

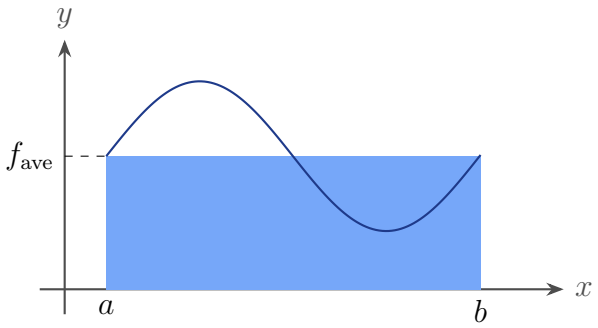
(2) if $f \geq g$ on R, then $\displaystyle\iint_R f\,\mathrm{d}A \geq \iint_R g\,\mathrm{d}A$.

If f takes both signs, then $\iint_R f\,\mathrm{d}A = V_1 - V_2$, where V_1 is the volume between the graph and the plane $z = 0$ where $f \geq 0$, and V_2 the same where $f \leq 0$ (signed volume, as for the area under a curve).

2.1.2 Average Value of a Function

For $f : [a, b] \to \mathbb{R}$ continuous, the average value f_{ave} is the number such that the rectangle of height f_{ave} has the same area as the region under the graph:

$$f_{\text{ave}}\,(b - a) = \int_a^b f(x)\,\mathrm{d}x.$$



Definition – Average Value over a Region

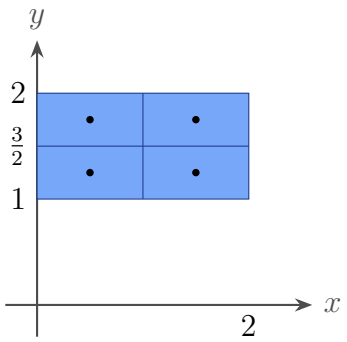
For f continuous on a region D of the plane,

$$f_{\text{ave}} = \frac{1}{\text{Area}(D)} \iint_D f(x, y)\,\mathrm{d}A.$$

For a rectangle $R = [a, b] \times [c, d]$ this reads $f_{\text{ave}}\,(b - a)(d - c) = \iint_R f\,\mathrm{d}A$: the volume of a box equals the volume under the graph. Examples: the average height of a mountain range, the average temperature over a region.

2.1.3 Iterated Integrals and Fubini's Theorem

Approximation by a Riemann sum. Let $f(x,y) = x - 3y^2$ on $R = [0,2] \times [1,2]$ with $m = n = 2$ and midpoints $(\frac{1}{2}, \frac{5}{4})$, $(\frac{1}{2}, \frac{7}{4})$, $(\frac{3}{2}, \frac{5}{4})$, $(\frac{3}{2}, \frac{7}{4})$. Here $\Delta x = 1$, $\Delta y = \frac{1}{2}$, $\Delta A = \frac{1}{2}$.



$$\iint_R f \, dA \simeq \sum_{i=1}^{2} \sum_{j=1}^{2} f(\bar{x}_i, \bar{y}_j) \, \Delta A = \tfrac{1}{2}\big(-4.1875 - 8.6875 - 3.1875 - 7.6875\big) = -11.875.$$

This is laborious and only approximate. Iterated integrals give the exact value.

Partial integration. For f on $[a,b] \times [c,d]$, fix y (treat it as a constant) and integrate in x. The result $\int_a^b f(x,y)\,dx$ depends on y, so we integrate it again, now in y:

$$\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_c^d \left[\int_a^b f(x,y)\,dx \right] dy \qquad \text{(iterated integral)}.$$

Theorem – Fubini's Theorem

If f is continuous on $R = [a,b] \times [c,d]$, then

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx.$$

Interpretation. Fix x and let $A(x) = \int_c^d f(x,y)\,dy$ be the area of the vertical slice of S at x. Then Volume $= \int_a^b A(x)\,dx$ (slicing method); the other order slices in the other direction.

Example 1 – Exact Value of the Riemann-Sum Example

$f(x,y) = x - 3y^2$ on $R = [0,2] \times [1,2]$. First

$$\int_0^2 (x - 3y^2)\,dx = \left[\tfrac{1}{2}x^2 - 3xy^2 \right]_{x=0}^{x=2} = 2 - 6y^2,$$

then

$$\int_1^2 (2 - 6y^2)\,dy = \left[2y - 2y^3 \right]_{y=1}^{y=2} = (4 - 16) - (2 - 2) = -12.$$

By Fubini, $\iint_R f\,dA = -12$, close to the approximation -11.875.

Example 2 – A Product with a Sine

$f(x,y) = x\sin y$, $R = [0,2]\times[0,\tfrac{\pi}{2}]$:

$$\iint_R f\,\mathrm{d}A = \int_0^2 x\,[-\cos y]_{y=0}^{y=\pi/2}\,\mathrm{d}x = \int_0^2 x\,\mathrm{d}x = 2.$$

Example 3 – A Rational Integrand

$f(x,y) = \dfrac{1}{(x+y)^2}$, $R = [0,1]\times[1,2]$. With $u = x+y$ in the inner integral,

$$\int_0^1 \frac{\mathrm{d}x}{(x+y)^2} = \Big[-\frac{1}{x+y}\Big]_{x=0}^{x=1} = \frac{1}{y} - \frac{1}{y+1},$$

$$\int_1^2 \Big(\frac{1}{y} - \frac{1}{y+1}\Big)\,\mathrm{d}y = \Big[\ln y - \ln(y+1)\Big]_1^2 = (\ln 2 - \ln 3) - (0 - \ln 2) = \ln\frac{4}{3}.$$

Example 4 – Choosing the Order

$f(x,y) = x\cos(xy)$, $R = [0,1]^2$. Integrating in y first is convenient:

$$\int_0^1\int_0^1 x\cos(xy)\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 \sin x\,\mathrm{d}x = 1 - \cos 1.$$

Integrating in x first would require an integration by parts. The order never changes the value, but it can change the work.

Property – Separable Integrands

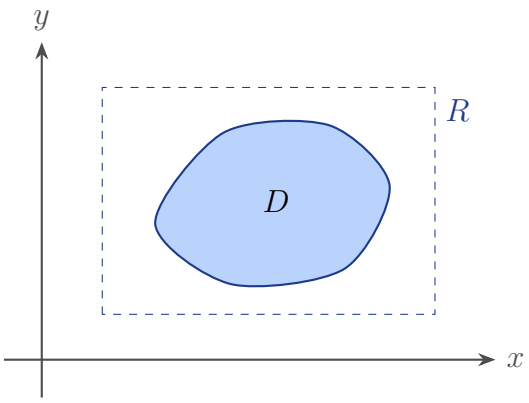
If $f(x,y) = g(x)h(y)$ and $R = [a,b]\times[c,d]$, then

$$\iint_R f\,\mathrm{d}A = \Big(\int_a^b g(x)\,\mathrm{d}x\Big)\Big(\int_c^d h(y)\,\mathrm{d}y\Big),$$

because the inner integral does not depend on y.

2.1.4 Double Integrals over General Regions

Let f be defined on a *bounded* region D, that is, D is contained in some rectangle R.



Definition – Integral over a Bounded Region

Choose a rectangle $R \supset D$ and extend f by zero:

$$F(x,y) = \begin{cases} f(x,y) & (x,y) \in D, \\ 0 & (x,y) \in R \setminus D. \end{cases} \qquad \iint_D f \, \mathrm{d}A := \iint_R F \, \mathrm{d}A.$$

How the integral is computed depends on the **shape** of D.

Definition – Type I and Type II Regions

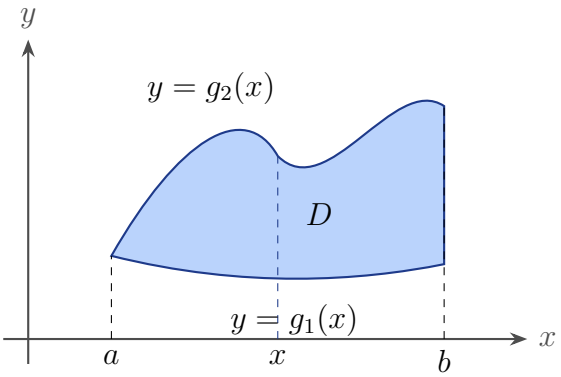
A region D is of **Type I** if it lies between the graphs of two continuous functions of x:

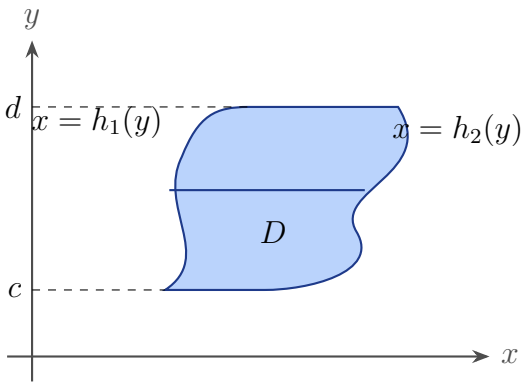
$$D = \{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}.$$

It is of **Type II** if it lies between the graphs of two continuous functions of y:

$$D = \{(x,y) \mid c \le y \le d,\ h_1(y) \le x \le h_2(y)\}.$$

Recall that a curve is the graph of a function of x only if each vertical line meets it at most once (and similarly with horizontal lines for a function of y).





Theorem – Fubini's Theorem on Type I and Type II Regions

Let f be continuous on D. If D is of Type I, fix x first (integrate in y), then integrate in x:

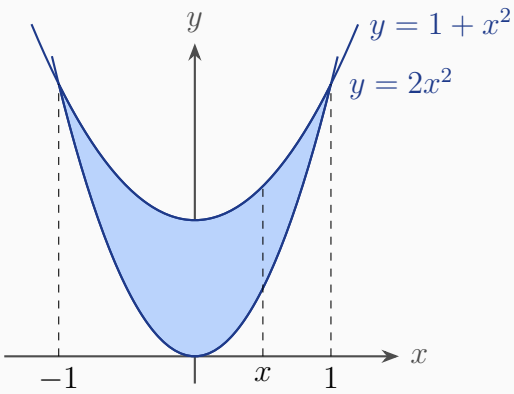
$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx.$$

If D is of Type II, fix y first:

$$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy.$$

Example 5 – A Type I Region

$f(x,y) = x + 2y$ on $D = \{-1 \le x \le 1,\ 2x^2 \le y \le 1 + x^2\}$.

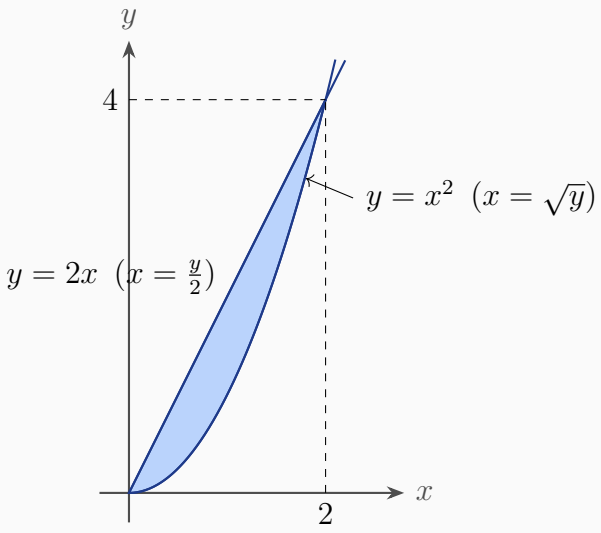


$$\int_{-1}^{1}\int_{2x^2}^{1+x^2} (x+2y)\,dy\,dx = \int_{-1}^{1} \Big[xy + y^2\Big]_{y=2x^2}^{y=1+x^2} dx = \int_{-1}^{1} \left(1 + x + 2x^2 - x^3 - 3x^4\right) dx.$$

The odd terms x and $-x^3$ integrate to zero, so the value is $2\left(1 + \frac{2}{3} - \frac{3}{5}\right) = \dfrac{32}{15}$.

Example 6 – A Region of Both Types

$D = \{0 \le x \le 2,\ x^2 \le y \le 2x\}$, which can also be written $D = \{0 \le y \le 4,\ \frac{y}{2} \le x \le \sqrt{y}\}$.



For any continuous f,

$$\iint_D f \, \mathrm{d}A = \int_0^2 \int_{x^2}^{2x} f(x,y) \, \mathrm{d}y \, \mathrm{d}x = \int_0^4 \int_{y/2}^{\sqrt{y}} f(x,y) \, \mathrm{d}x \, \mathrm{d}y.$$

Depending on f, one of the two iterated integrals may be much simpler, for instance when f depends on only one of the variables.

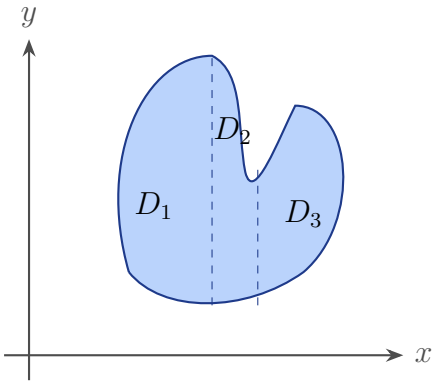
Remark

A region can be of both types (a disk, a lens), of one type only, or of neither. A region with a dent on one side is Type I only or Type II only, depending on the direction of the dent. A region with dents on two sides is neither.

Splitting a region. If $D = D_1 \cup D_2 \cup \cdots \cup D_k$ and the pieces overlap only along their boundaries, then

$$\iint_D f \, \mathrm{d}A = \sum_{i=1}^{k} \iint_{D_i} f \, \mathrm{d}A.$$

This handles regions that are neither Type I nor Type II: cut them into pieces that are.



Property – Properties over a Region

For integrable f, g on D:

(1) **Linearity:** $\iint_D (\alpha f + \beta g)\,\mathrm{d}A = \alpha \iint_D f\,\mathrm{d}A + \beta \iint_D g\,\mathrm{d}A$.

(2) **Monotonicity:** if $f \le g$ on D, then $\iint_D f\,\mathrm{d}A \le \iint_D g\,\mathrm{d}A$.

(3) **Area:** $\iint_D 1\,\mathrm{d}A = \mathrm{Area}(D)$ (the solid of base D and height 1).

(4) **Bounds:** if $m \le f \le M$ on D, then $m\,\mathrm{Area}(D) \le \iint_D f\,\mathrm{d}A \le M\,\mathrm{Area}(D)$.

(5) **Symmetry:** if D is symmetric about the y-axis and $f(-x, y) = -f(x, y)$, then $\iint_D f\,\mathrm{d}A = 0$; similar statements hold for the x-axis and the origin.

Definition – Area and Volume

For a planar region D, $\mathrm{Area}(D) = \iint_D 1\,\mathrm{d}A$. If $f \ge 0$ on D, the volume under $z = f(x, y)$ above D is $V = \iint_D f(x, y)\,\mathrm{d}A$.

Example 7 – Rectangle and Triangle

(a) $\iint_D xy^2\,\mathrm{d}A$ over $D = [0, 2] \times [0, 1]$: the integrand separates, so the value is $\int_0^2 x\,\mathrm{d}x \int_0^1 y^2\,\mathrm{d}y = 2 \cdot \frac{1}{3} = \frac{2}{3}$.
(b) $\iint_D (x + y)\,\mathrm{d}A$ over the triangle $D = \{0 \le x \le 1,\ 0 \le y \le x\}$:

$$\int_0^1 \int_0^x (x + y)\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 \left[xy + \frac{y^2}{2}\right]_0^x \mathrm{d}x = \int_0^1 \frac{3x^2}{2}\,\mathrm{d}x = \frac{1}{2}.$$

(c) Area of $D = \{0 \le x \le 1,\ 0 \le y \le 1 - x\}$: $\int_0^1 (1 - x)\,\mathrm{d}x = \frac{1}{2}$.

2.1.5 Changing the Order of Integration

Proposition – Reversing the Order

To reverse $\int_a^b \int_{\alpha(x)}^{\beta(x)} f(x, y)\,\mathrm{d}y\,\mathrm{d}x$: (1) read off the region D from the limits; (2) sketch it; (3) describe the same region as a Type II region and read the new limits. The value is unchanged when the hypotheses of Fubini's theorem hold. Identifying the region is the safest way to avoid mistakes.

Example 8 – Reversing the Order

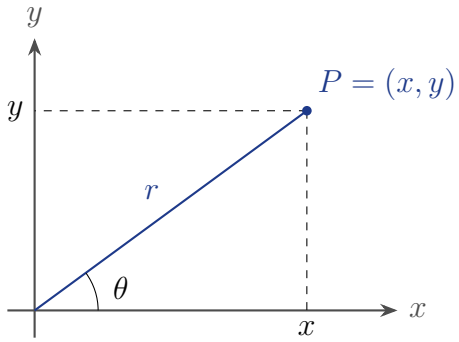
$I = \int_0^1 \int_x^1 (x+y)\,dy\,dx$. The region is $0 \le x \le y \le 1$, so

$$I = \int_0^1 \int_0^y (x+y)\,dx\,dy = \int_0^1 \left(\frac{y^2}{2} + y^2\right) dy = \frac{1}{2}.$$

2.1.6 Polar Coordinates

2.1.6.1 Polar coordinates in the plane

A point $P = (x, y)$ has polar coordinates (r, θ): r is its distance to the origin and θ the angle from the positive x-axis.



Property – Conversion Formulas

$$x = r\cos\theta, \quad y = r\sin\theta \qquad \text{and} \qquad r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x}.$$

Remark

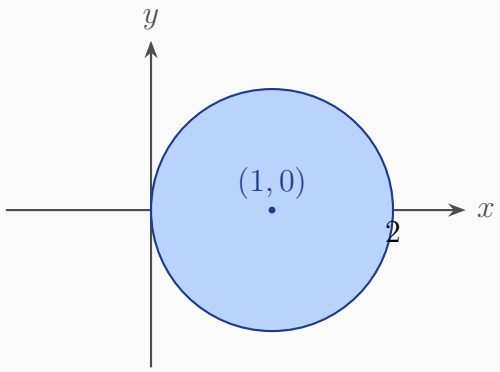
A point has several polar representations. For $(1, 1)$, whose distance to O is $\sqrt{2}$, we may take $(\sqrt{2}, \frac{\pi}{4})$, $(\sqrt{2}, \frac{\pi}{4} + 2\pi)$, or $(-\sqrt{2}, \frac{\pi}{4} + \pi)$ if negative r is allowed.

A **polar curve** is given by $r = r(\theta)$, $\theta \in [\theta_0, \theta_1]$. It is the parametric curve $x(\theta) = r(\theta)\cos\theta$, $y(\theta) = r(\theta)\sin\theta$.

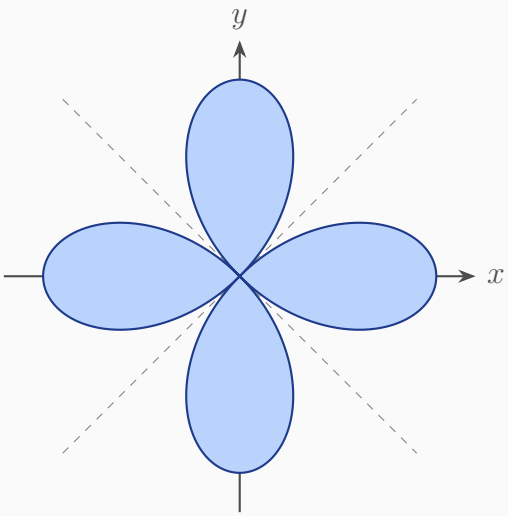
Example 9 – Three Polar Curves

(a) $r = 1$: the unit circle.

(b) $r = 2\cos\theta$ (note that $r < 0$ for some θ). Instead of plotting points, convert: $x = r\cos\theta = \frac{r^2}{2}$, so $r^2 = 2x$, that is $x^2 + y^2 = 2x$, or $(x-1)^2 + y^2 = 1$: a circle of center $(1, 0)$ and radius 1.



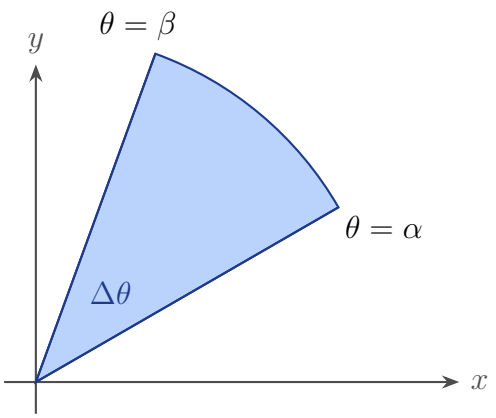
(c) $r = \cos(2\theta)$: a four-leaved rose, since the graph of $r(\theta)$ on $[0, 2\pi]$ has eight alternating-sign arcs.



2.1.6.2 Areas in polar coordinates

The area of a circular sector of radius r and angle θ is $\frac{1}{2}r^2\theta$. For a polar curve $r : [\alpha, \beta] \to \mathbb{R}$, cut $[\alpha, \beta]$ into n pieces of width $\Delta\theta$, choose θ_i^* in each, and add the areas of the sectors:

$$A \simeq \sum_{i=1}^{n} \tfrac{1}{2}\,[r(\theta_i^*)]^2\,\Delta\theta.$$



Letting $n \to \infty$ gives the area formula.

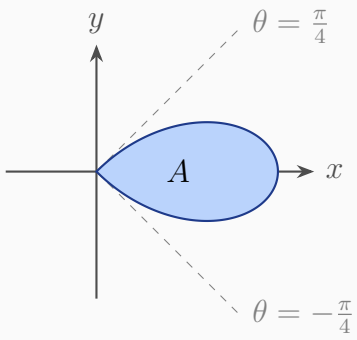
Theorem – Area in Polar Coordinates

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2(\theta)\,\mathrm{d}\theta.$$

For the region between two polar curves $0 \le g \le f$ on $[\alpha, \beta]$: $A = \frac{1}{2}\int_{\alpha}^{\beta}\left[f^2(\theta) - g^2(\theta)\right]\mathrm{d}\theta$.

Example 10 – Area of a Petal of $r = \cos 2\theta$

One petal corresponds to $\theta \in [-\frac{\pi}{4}, \frac{\pi}{4}]$.



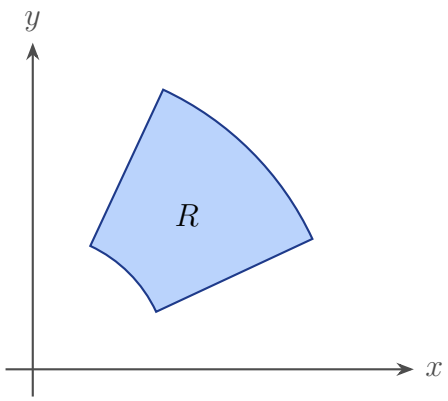
By symmetry and $\cos^2(2\theta) = \frac{1}{2}\big(1 + \cos 4\theta\big)$,

$$A = \frac{1}{2}\int_{-\pi/4}^{\pi/4} \cos^2(2\theta)\,\mathrm{d}\theta = \int_{0}^{\pi/4} \frac{1+\cos(4\theta)}{2}\,\mathrm{d}\theta = \frac{\pi}{8} + \left[\frac{\sin 4\theta}{8}\right]_0^{\pi/4} = \frac{\pi}{8}.$$

2.1.6.3 Double integrals in polar coordinates

Definition – Polar Rectangle

A **polar rectangle** is a region $R = \{(r, \theta) \mid r_1 \le r \le r_2,\ \alpha \le \theta \le \beta\}$ with $r_1 \ge 0$.



Theorem – Double Integral in Polar Coordinates

If f is continuous on the polar rectangle R, then

$$\iint_R f(x,y)\,\mathrm{d}A = \int_\alpha^\beta \int_{r_1}^{r_2} f(r\cos\theta, r\sin\theta)\, r\,\mathrm{d}r\,\mathrm{d}\theta \qquad \text{(note the extra factor } r\text{!)}.$$

More generally, for $D = \{\alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\}$,

$$\iint_D f\,\mathrm{d}A = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\,\mathrm{d}r\,\mathrm{d}\theta.$$

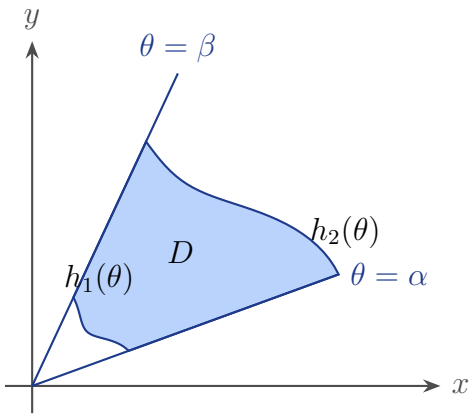
Fix θ, integrate in r, then integrate in θ.

Idea of proof. Cut R into polar sub-rectangles R_{ij} with $\Delta r = \frac{r_2 - r_1}{m}$ and $\Delta\theta = \frac{\beta-\alpha}{n}$. Their areas are *not* equal: the difference of two sectors gives

$$\Delta A_{ij} = \tfrac{1}{2}(r_i^2 - r_{i-1}^2)\Delta\theta = \tfrac{1}{2}(r_i + r_{i-1})(r_i - r_{i-1})\Delta\theta = r_i^*\,\Delta r\,\Delta\theta,$$

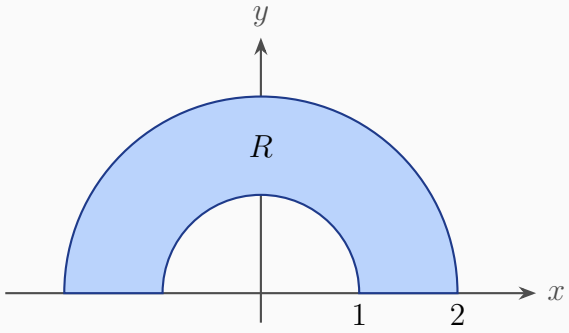
with r_i^* the midpoint radius. Hence the Riemann sums are

$$\sum_{i,j} f(r_i^*\cos\theta_j^*,\ r_i^*\sin\theta_j^*)\, r_i^*\,\Delta r\,\Delta\theta \xrightarrow[m,n\to\infty]{} \int_\alpha^\beta \int_{r_1}^{r_2} f(r\cos\theta, r\sin\theta)\, r\,\mathrm{d}r\,\mathrm{d}\theta. \qquad \blacksquare$$



Example 11 – A Half Annulus

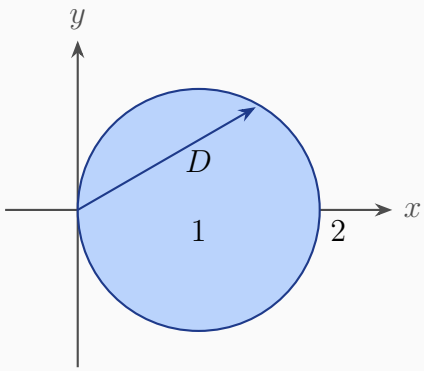
$f(x,y) = 3x + 4y^2$ over the upper half of $1 \le x^2 + y^2 \le 4$, i.e. $1 \le r \le 2$, $0 \le \theta \le \pi$.



$$\iint_R f\,dA = \int_0^{\pi}\int_1^2 (3r^2\cos\theta + 4r^3\sin^2\theta)\,dr\,d\theta = \int_0^{\pi}\left(7\cos\theta + 15\sin^2\theta\right)d\theta$$
$$= \int_0^{\pi}\left[7\cos\theta + \tfrac{15}{2}(1-\cos 2\theta)\right]d\theta = \frac{15\pi}{2}.$$

Example 12 – A Disk Not Centered at the Origin

Integrate $f(x,y) = x^2 + y^2$ over the disk $(x-1)^2 + y^2 \le 1$.



The boundary is $r = 2\cos\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ (see the previous example of curves). Then

$$\iint_D (x^2+y^2)\,dA = \int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r^2\, r\,dr\,d\theta = 4\int_{-\pi/2}^{\pi/2}\cos^4\theta\,d\theta = 8\int_0^{\pi/2}\cos^4\theta\,d\theta.$$

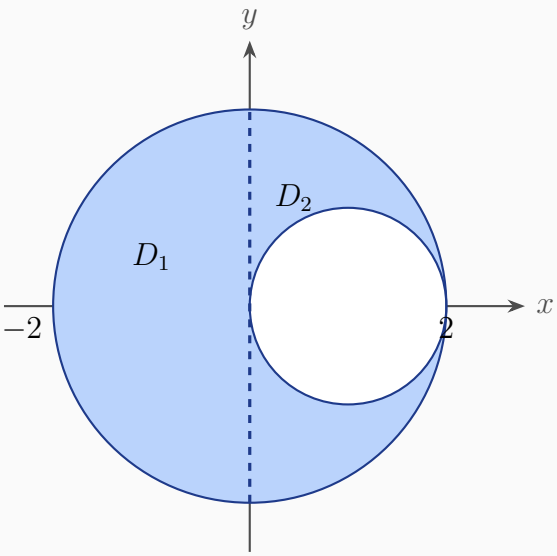
Using $\cos^4\theta = \frac{1}{4}\left(\frac{3}{2} + 2\cos 2\theta + \frac{1}{2}\cos 4\theta\right)$,

$$8\int_0^{\pi/2}\cos^4\theta\,d\theta = 2\left[\tfrac{3}{2}\theta + \sin 2\theta + \tfrac{1}{8}\sin 4\theta\right]_0^{\pi/2} = \frac{3\pi}{2}.$$

In Cartesian coordinates the same integral is far harder.

Example 13 – A Disk with a Hole

Integrate $f(x,y) = x^2 + y^2$ over D, the disk $x^2 + y^2 \le 4$ with the disk $(x-1)^2 + y^2 \le 1$ removed. Cut D along the y-axis into $D_1 \cup D_2$.



D_1 is the left half-disk ($\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$, $0 \le r \le 2$) and D_2 is the right half-disk minus the small disk ($-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, $2\cos\theta \le r \le 2$).

$$\iint_{D_1} f \, \mathrm{d}A = \int_{\pi/2}^{3\pi/2} \int_0^2 r^3 \, \mathrm{d}r \, \mathrm{d}\theta = 4\pi,$$

$$\iint_{D_2} f \, \mathrm{d}A = \int_{-\pi/2}^{\pi/2} \int_{2\cos\theta}^2 r^3 \, \mathrm{d}r \, \mathrm{d}\theta = 4 \int_{-\pi/2}^{\pi/2} \left(1 - \cos^4\theta\right) \mathrm{d}\theta = 4\pi - \frac{3\pi}{2} = \frac{5\pi}{2}.$$

Hence $\iint_D f \, \mathrm{d}A = 4\pi + \frac{5\pi}{2} = \dfrac{13\pi}{2}$. (Check: the full disk gives 8π, and $8\pi - \frac{3\pi}{2} = \frac{13\pi}{2}$.)

Example 14 – Gaussian over the Unit Disk

$\iint_D e^{-(x^2+y^2)} \, \mathrm{d}A$ for $D: \ x^2 + y^2 \le 1$:

$$\int_0^{2\pi} \int_0^1 e^{-r^2} \, r \, \mathrm{d}r \, \mathrm{d}\theta = 2\pi \Big[-\tfrac{1}{2} e^{-r^2} \Big]_0^1 = \pi\left(1 - e^{-1}\right).$$

2.1.7 Change of Variables and the Jacobian

Definition – Jacobian

For a differentiable transformation $x = x(u, v)$, $y = y(u, v)$, the **Jacobian determinant** is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}.$$

Its absolute value is the local factor by which the transformation scales areas.

Theorem – Change of Variables Formula

Let $T(u,v) = (x(u,v), y(u,v))$ be a one-to-one transformation with continuous partial derivatives and nonzero Jacobian, mapping a region D^* onto D. Then

$$\iint_D f(x,y)\,\mathrm{d}A = \iint_{D^*} f\big(x(u,v), y(u,v)\big)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| \mathrm{d}u\,\mathrm{d}v.$$

Proposition – Polar Coordinates as a Special Case

For $x = r\cos\theta$, $y = r\sin\theta$,

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix}\cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta\end{vmatrix} = r, \qquad \text{so} \qquad \mathrm{d}A = r\,\mathrm{d}r\,\mathrm{d}\theta.$$

Circular and radial regions become rectangles in the (r,θ)-plane.

Example 15 – Area of an Annulus

For the annulus $a^2 \le x^2 + y^2 \le b^2$ with $0 \le a < b$,

$$A = \int_0^{2\pi}\int_a^b r\,\mathrm{d}r\,\mathrm{d}\theta = 2\pi\left[\frac{r^2}{2}\right]_a^b = \pi\big(b^2 - a^2\big).$$

Example 16 – A Linear Change of Variables

Compute $\iint_D (x+y)(x-y)\,\mathrm{d}A$, where D is the parallelogram bounded by $x+y=1$, $x+y=3$, $x-y=0$, $x-y=2$. Put $u = x+y$, $v = x-y$, so $x = \frac{u+v}{2}$, $y = \frac{u-v}{2}$ and

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix}\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2}\end{vmatrix} = -\frac{1}{2}.$$

Then $D^* = [1,3]\times[0,2]$ and

$$\iint_D (x+y)(x-y)\,\mathrm{d}A = \frac{1}{2}\int_1^3 u\,\mathrm{d}u \int_0^2 v\,\mathrm{d}v = \frac{1}{2}\cdot 4\cdot 2 = 4.$$

2.2 Triple Integrals

Triple integrals extend integration to 3D regions $V \subset \mathbb{R}^3$:

$$\iiint_V f(x,y,z)\,dV.$$

Proposition – Cylindrical Coordinates

With $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, the volume element is $dV = r\,dr\,d\theta\,dz$:

$$\iiint_V f\,dV = \iiint_{V^*} f(r\cos\theta, r\sin\theta, z)\,r\,dr\,d\theta\,dz.$$

Proposition – Spherical Coordinates

With $x = \rho\sin\varphi\cos\theta$, $y = \rho\sin\varphi\sin\theta$, $z = \rho\cos\varphi$, the volume element is $dV = \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$:

$$\iiint_V f\,dV = \iiint_{V^*} f(\rho,\varphi,\theta)\,\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta.$$

Example 1 – Volume of a Ball via Spherical Coordinates

The volume of the ball of radius R:

$$V = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta = 2\pi\cdot 2\cdot\frac{R^3}{3} = \frac{4}{3}\pi R^3.$$

2.3 Triple Integrals and Applications

Definition – Triple Integral

If f is integrable over a three-dimensional region V, its triple integral is denoted

$$\iiint_V f(x,y,z)\,dV.$$

For $f = 1$, it gives the volume of V.

Property – Separability on Rectangular Boxes

If $V = [a,b]\times[c,d]\times[e,h]$ and $f(x,y,z) = p(x)q(y)r(z)$, then

$$\iiint_V f\,dV = \left(\int_a^b p(x)\,dx\right)\left(\int_c^d q(y)\,dy\right)\left(\int_e^h r(z)\,dz\right).$$

Example 1 – Mass of a Box

A box $0\le x\le 1$, $0\le y\le 2$, $0\le z\le 3$ has density $\rho(x,y,z) = x+y+z$. Its mass is

$$M = \int_0^1\int_0^2\int_0^3 (x+y+z)\,dz\,dy\,dx.$$

Integrating term by term gives

$$M = 9 + 6 + 9 = 24.$$

2.3.1 Further Applications of Multiple Integration

Definition – Center of Mass

For a lamina occupying D with density $\rho(x, y)$, its mass is

$$M = \iint_D \rho \, dA.$$

The coordinates of its center of mass are

$$\bar{x} = \frac{1}{M} \iint_D x\rho \, dA, \qquad \bar{y} = \frac{1}{M} \iint_D y\rho \, dA.$$

Definition – Moment of Inertia

The moment of inertia of a planar lamina about the x-axis is

$$I_x = \iint_D y^2 \rho(x, y) \, dA,$$

and about the y-axis is

$$I_y = \iint_D x^2 \rho(x, y) \, dA.$$

Example 2 – Center of Mass of a Uniform Triangle

For the triangle

$$D = \{(x, y) : x \geq 0, \ y \geq 0, \ x + y \leq 1\}$$

with constant density, symmetry and direct integration give

$$M = \frac{1}{2}, \qquad \iint_D x \, dA = \frac{1}{6}, \qquad \iint_D y \, dA = \frac{1}{6}.$$

Thus

$$\bar{x} = \bar{y} = \frac{1/6}{1/2} = \frac{1}{3}.$$

Theorem – Change of Variables in Three Dimensions

If

$$(x, y, z) = T(u, v, w)$$

is a sufficiently smooth one-to-one transformation with nonzero Jacobian, then

$$\iiint_V f(x, y, z)\, dV = \iiint_{V^*} f(T(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du\, dv\, dw.$$

2.4 Worked Exercises

Exercise 1. Average Value. Find the average value of $f(x, y) = x - 3y^2$ over $R = [0, 2] \times [1, 2]$.

Solution. Area$(R) = 2$ and $\iint_R f \, \mathrm{d}A = -12$ (see the example above), so $f_{\text{ave}} = -12/2 = -6$.

Exercise 2. Reversing the Order. Compute $I = \int_0^1 \int_x^1 e^{y^2} \, \mathrm{d}y \, \mathrm{d}x$.

Solution. The function e^{y^2} has no elementary antiderivative, so reverse the order. The region is $0 \le x \le y \le 1$:

$$I = \int_0^1 \int_0^y e^{y^2} \, \mathrm{d}x \, \mathrm{d}y = \int_0^1 y\, e^{y^2} \, \mathrm{d}y = \left[\tfrac{1}{2} e^{y^2} \right]_0^1 = \frac{e - 1}{2}.$$

Exercise 3. Volume under a Paraboloid. Find the volume under $z = 4 - x^2 - y^2$ above the disk $x^2 + y^2 \le 1$.

Solution. In polar coordinates,

$$V = \int_0^{2\pi} \int_0^1 (4 - r^2)\, r \, \mathrm{d}r \, \mathrm{d}\theta = 2\pi \left[2r^2 - \frac{r^4}{4} \right]_0^1 = \frac{7\pi}{2}.$$

Exercise 4. Integral over a Disk. Evaluate $I = \iint_D (x^2 + y^2) \, \mathrm{d}A$ over $D: \; x^2 + y^2 \le 4$.

Solution. $I = \int_0^{2\pi} \int_0^2 r^3 \, \mathrm{d}r \, \mathrm{d}\theta = 2\pi \left[\frac{r^4}{4} \right]_0^2 = 8\pi$.

Exercise 5. Triple Integral over a Region. Compute $\iiint_V z \, dV$ where V is the region bounded by $z = 0$, $z = 1$, and the cylinder $x^2 + y^2 \le 1$.

Solution. Using cylindrical coordinates,

$$\iiint_V z \, dV = \int_0^{2\pi} \int_0^1 \int_0^1 z\, r \, dz \, dr \, d\theta = 2\pi \cdot \left(\int_0^1 r \, dr \right) \left(\int_0^1 z \, dz \right) = 2\pi \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{\pi}{2}.$$

2.4.1 Additional Worked Exercises

Exercise 6. Triple Integral in Spherical Coordinates. Find the volume of the ball $x^2 + y^2 + z^2 \le R^2$.

Solution. In spherical coordinates,

$$V = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta = 2\pi \cdot 2 \cdot \frac{R^3}{3} = \frac{4\pi R^3}{3}.$$

Exercise 7. Center of Mass. Find the center of mass of the unit square $0 \le x, y \le 1$ with constant density.

Solution. The mass is $M = 1$. Therefore

$$\bar{x} = \int_0^1\int_0^1 x \, dy \, dx = \int_0^1 x \, dx = \frac{1}{2},$$

and similarly $\bar{y} = 1/2$. Hence the center of mass is $(1/2, 1/2)$.

Exercise 8. Volume Between Surfaces. Find the volume under $z = 1 - x^2 - y^2$ and above the xy-plane.

Solution. The projection is the unit disk. In polar coordinates,

$$V = \int_0^{2\pi}\int_0^1 (1 - r^2) r \, dr \, d\theta = 2\pi \left[\frac{r^2}{2} - \frac{r^4}{4}\right]_0^1 = \frac{\pi}{2}.$$

2.4.2 Review Exercises

In two and three dimensions, the principal difficulty is often the description of the region rather than the integration itself. Students should be able to sketch a region, choose an order of integration, recognize when polar, cylindrical, or spherical coordinates simplify the geometry, and compute the appropriate Jacobian. For improper integrals, always identify the source of impropriety and replace it by a limit before applying convergence tests.

Exercise 9. Polar Coordinates. Evaluate $\iint_D 1 \, dA$ for the disk $x^2 + y^2 \le 9$.

Solution. In polar coordinates,

$$\iint_D 1 \, dA = \int_0^{2\pi}\int_0^3 r \, dr \, d\theta = 2\pi \frac{9}{2} = 9\pi.$$

Exercise 10. Polar Coordinates. Evaluate $\iint_D (x^2 + y^2) \, dA$, where D is the unit disk.

Solution.

$$I = \int_0^{2\pi}\int_0^1 r^2 r \, dr \, d\theta = 2\pi \frac{1}{4} = \frac{\pi}{2}.$$

2.4.3 Supplementary Exercises

Exercise 2. Double integrals

1) Evaluate:

a) $\displaystyle\int_0^2\int_0^1 xy^2 \, \mathrm{d}x \, \mathrm{d}y$

Solution. The integrand separates: $\displaystyle\int_0^2\int_0^1 xy^2\,dx\,dy = \left(\int_0^1 x\,dx\right)\left(\int_0^2 y^2\,dy\right) = \frac{1}{2}\cdot\frac{8}{3} = \frac{4}{3}$.

b) $\displaystyle\int_0^{\ln 4}\int_0^{\ln 3} e^{x+y}\,dx\,dy$

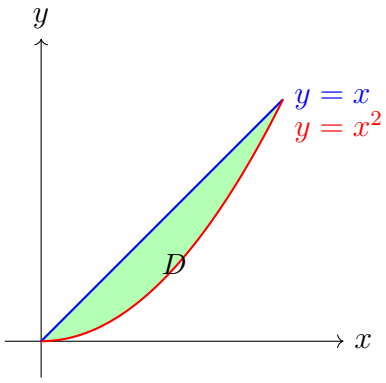
Solution. $\displaystyle\int_0^{\ln 4}\int_0^{\ln 3} e^x e^y\,dx\,dy = \left(\int_0^{\ln 3} e^x\,dx\right)\left(\int_0^{\ln 4} e^y\,dy\right) = (3-1)(4-1) = 6.$

c) $\displaystyle\int_0^1\int_y^1 e^{x^2}\,dx\,dy$

Solution. This iterated integral cannot be done directly (e^{x^2} has no elementary antiderivative), so we *swap the order of integration*. The domain is $0 \le y \le x \le 1$, so

$$\int_0^1\int_y^1 e^{x^2}\,dx\,dy = \int_0^1\int_0^x e^{x^2}\,dy\,dx = \int_0^1 x\,e^{x^2}\,dx = \left[\tfrac{1}{2}e^{x^2}\right]_0^1 = \frac{e-1}{2}.$$

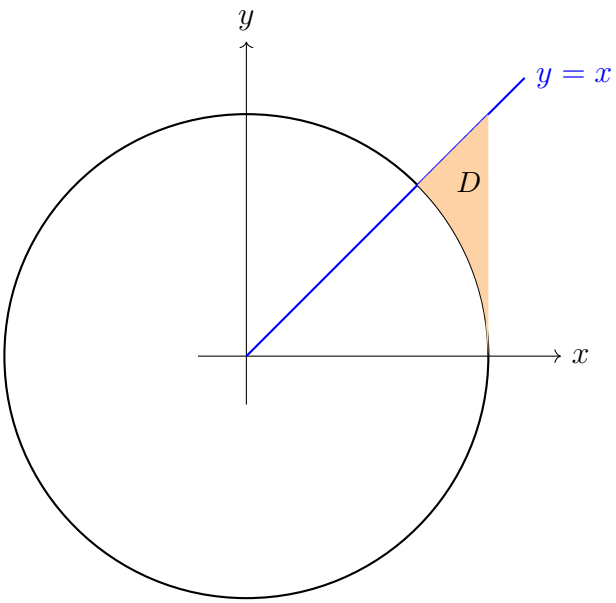
2) Region bounded by $y = x$ and $y = x^2$: evaluate $\displaystyle\iint_D x\,dA$.



Solution. On $[0,1]$, $x^2 \le x$, so $D = \{(x,y) : 0 \le x \le 1,\ x^2 \le y \le x\}$ (a lens-shaped region between the parabola and the line, tangent at $(0,0)$ and $(1,1)$).

$$\iint_D x\,dA = \int_0^1\int_{x^2}^x x\,dy\,dx = \int_0^1 x(x-x^2)\,dx = \int_0^1 (x^2-x^3)\,dx = \frac{1}{3}-\frac{1}{4} = \frac{1}{12}.$$

3) Area of $D = \left\{(x,y)\in\mathbb{R}^2 : \sqrt{1-x^2} \le y \le x \le 1\right\}$.



Solution. The condition $\sqrt{1-x^2} \le x$ forces $1-x^2 \le x^2$, i.e. $x \ge \frac{1}{\sqrt{2}}$ (with $x \ge 0$). So D lies between the unit circle and the line $y = x$, for $x \in \left[\frac{1}{\sqrt{2}}, 1\right]$:

$$\text{Area} = \int_{1/\sqrt{2}}^{1} \left(x - \sqrt{1-x^2}\right) \mathrm{d}x.$$

First piece: $\displaystyle\int_{1/\sqrt{2}}^{1} x \,\mathrm{d}x = \left[\tfrac{x^2}{2}\right]_{1/\sqrt{2}}^{1} = \tfrac{1}{2} - \tfrac{1}{4} = \tfrac{1}{4}.$

Second piece: using $\int \sqrt{1-x^2}\,\mathrm{d}x = \frac{1}{2}\big(x\sqrt{1-x^2} + \arcsin x\big)$,

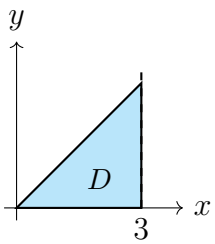
$$\int_{1/\sqrt{2}}^{1} \sqrt{1-x^2}\,\mathrm{d}x = \frac{1}{2}\Big(0 + \frac{\pi}{2}\Big) - \frac{1}{2}\Big(\frac{1}{2} + \frac{\pi}{4}\Big) = \frac{\pi}{4} - \frac{1}{4} - \frac{\pi}{8} = \frac{\pi}{8} - \frac{1}{4}.$$

Hence

$$\text{Area} = \frac{1}{4} - \left(\frac{\pi}{8} - \frac{1}{4}\right) = \frac{1}{2} - \frac{\pi}{8} = \frac{4-\pi}{8}.$$

4) Using polar coordinates ($x = r\cos\theta,\ y = r\sin\theta,\ \mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\theta$)**:**

a) $D = \{0 \le x \le 3,\ 0 \le y \le x\}$ (a right triangle).

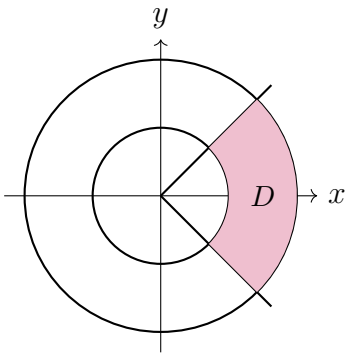


Solution. Here $\theta \in [0, \frac{\pi}{4}]$ (since $y \le x$), and the bound $x \le 3$ becomes $r\cos\theta \le 3$, i.e. $r \le 3/\cos\theta$. Thus

$$\iint_D \mathrm{d}x\,\mathrm{d}y = \int_0^{\pi/4} \int_0^{3/\cos\theta} r\,\mathrm{d}r\,\mathrm{d}\theta = \int_0^{\pi/4} \frac{1}{2}\cdot\frac{9}{\cos^2\theta}\,\mathrm{d}\theta = \frac{9}{2}\big[\tan\theta\big]_0^{\pi/4} = \frac{9}{2}.$$

(Check: the triangle has legs 3 and 3, area $= \frac{1}{2} \cdot 3 \cdot 3 = \frac{9}{2}$. ✓)

b) $D = \{x \geq 0,\ y^2 \leq x^2,\ 1 \leq x^2 + y^2 \leq 4\}$, evaluate $\displaystyle\iint_D \frac{1}{1 + \sqrt{x^2 + y^2}}\, \mathrm{d}x\, \mathrm{d}y$.



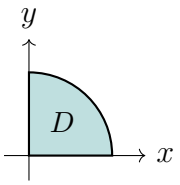
Solution. $y^2 \leq x^2$ with $x \geq 0$ means $-x \leq y \leq x$, i.e. $\theta \in [-\frac{\pi}{4}, \frac{\pi}{4}]$; the annulus gives $r \in [1, 2]$.

$$\iint_D \frac{r}{1+r}\, \mathrm{d}r\, \mathrm{d}\theta = \left(\int_{-\pi/4}^{\pi/4} \mathrm{d}\theta \right) \left(\int_1^2 \frac{r}{1+r}\, \mathrm{d}r \right) = \frac{\pi}{2} \int_1^2 \left(1 - \frac{1}{1+r} \right) \mathrm{d}r.$$

$$\int_1^2 \left(1 - \frac{1}{1+r} \right) \mathrm{d}r = \big[r - \ln(1+r) \big]_1^2 = (2 - \ln 3) - (1 - \ln 2) = 1 + \ln \frac{2}{3}.$$

So the integral equals $\displaystyle\frac{\pi}{2} \left(1 - \ln \frac{3}{2} \right)$.

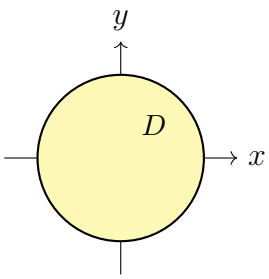
c) $D = \{(x, y) \in \mathbb{R}^2_+ : xy \geq 0,\ x^2 + y^2 \leq 1\}$, evaluate $\displaystyle\iint_D \frac{xy}{x^2 + y^2}\, \mathrm{d}x\, \mathrm{d}y$.



Solution. $\mathbb{R}^2_+$ means $x \geq 0, y \geq 0$ (the condition $xy \geq 0$ is then automatic); D is the quarter unit disk, $\theta \in [0, \frac{\pi}{2}]$, $r \in [0, 1]$. Since $\displaystyle\frac{xy}{x^2 + y^2} = \cos\theta \sin\theta = \frac{1}{2} \sin 2\theta$ (independent of r),

$$\iint_D \frac{xy}{x^2 + y^2}\, \mathrm{d}x\, \mathrm{d}y = \int_0^{\pi/2} \frac{1}{2} \sin 2\theta\ \mathrm{d}\theta \int_0^1 r\, \mathrm{d}r = \left[-\frac{1}{4} \cos 2\theta \right]_0^{\pi/2} \cdot \frac{1}{2} = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$$

d) D: unit disk, $\displaystyle\iint_D e^{\sqrt{x^2+y^2}}\, \mathrm{d}x\, \mathrm{d}y$.

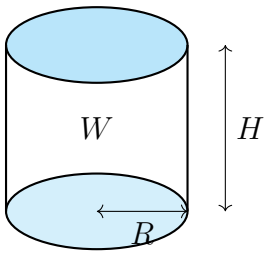


Solution. $\theta \in [0, 2\pi]$, $r \in [0, 1]$:

$$\iint_D e^r \, r \, \mathrm{d}r \, \mathrm{d}\theta = 2\pi \int_0^1 r \, e^r \, \mathrm{d}r = 2\pi \Big[(r-1)e^r \Big]_0^1 = 2\pi \big[0 - (-1) \big] = 2\pi.$$

Exercise 3. Triple integrals — volumes and mass

a) Cylinder of radius R and height H: $W = \{x^2 + y^2 \le R^2,\ 0 \le z \le H\}$.



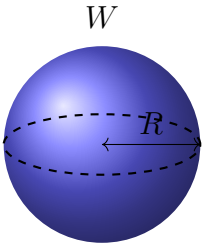
Solution. Cylindrical coordinates ($\mathrm{d}V = r \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}z$):

$$\iiint_W \mathrm{d}V = \int_0^{2\pi} \int_0^R \int_0^H r \, \mathrm{d}z \, \mathrm{d}r \, \mathrm{d}\theta = 2\pi \cdot H \cdot \frac{R^2}{2} = \pi R^2 H.$$

b) Ball of radius R.

The ball of radius R is obtained by bounding z on *both* sides:

$$W = \left\{ (x, y, z) \in \mathbb{R}^3 : \ -\sqrt{R^2 - x^2 - y^2} \ \le z \le \sqrt{R^2 - x^2 - y^2} \right\} = \left\{ x^2 + y^2 + z^2 \le R^2 \right\}.$$



Solution. Spherical coordinates ($x = \rho \sin\varphi \cos\theta$, $y = \rho \sin\varphi \sin\theta$, $z = \rho \cos\varphi$, $\mathrm{d}V =$

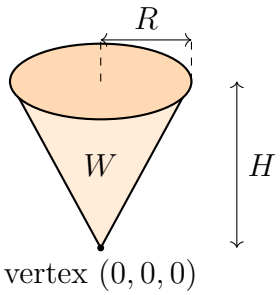
$\rho^2 \sin\varphi \; \mathrm{d}\rho\, \mathrm{d}\varphi\, \mathrm{d}\theta$), with $\rho \in [0, R]$, $\varphi \in [0, \pi]$, $\theta \in [0, 2\pi]$:

$$\iiint_W \mathrm{d}V = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} \rho^2 \sin\varphi \; \mathrm{d}\rho\, \mathrm{d}\varphi\, \mathrm{d}\theta = 2\pi \cdot \big[-\cos\varphi\big]_0^{\pi} \cdot \frac{R^3}{3} = 2\pi \cdot 2 \cdot \frac{R^3}{3} = \frac{4}{3}\pi R^3.$$

c) Cone of radius R and height H.

The cone of height H and circular base of radius R :

$$W = \Big\{(x, y, z) \in \mathbb{R}^3 : \; 0 \le z \le H, \;\; r^2 = x^2 + y^2 \le \Big(\frac{R}{H}\, z\Big)^2\Big\}.$$



Solution. In cylindrical coordinates, for fixed $z \in [0, H]$ the disk has radius $\frac{R}{H}z$:

$$\iiint_W \mathrm{d}V = \int_0^{H}\int_0^{2\pi}\int_0^{Rz/H} r \,\mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}z = \int_0^{H} 2\pi \cdot \frac{1}{2}\Big(\frac{R}{H}z\Big)^2 \mathrm{d}z = \pi\frac{R^2}{H^2}\int_0^{H} z^2\, \mathrm{d}z = \pi\frac{R^2}{H^2}\cdot\frac{H^3}{3} = \frac{1}{3}\pi R^2 H,$$

which is the familiar cone volume formula.

Indeed. Let us calculate the volume of a cone with radius R and height H, using cylindrical coordinates.

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z.$$

The volume element is

$$\boxed{dV = r\, dr\, dz\, d\theta.}$$

The factor r is the Jacobian of the transformation. Therefore, the volume of the cone is

$$V = \iiint_E r\, dr\, dz\, d\theta.$$

To find equation of the Cone, we consider a cone whose vertex is at the origin and whose base is in the plane $z = H$. The radius of the base is R. From similar triangles, we have

$$\frac{r}{z} = \frac{R}{H}.$$

Therefore,

$$r = \frac{R}{H}z$$

Equivalently,

$$\boxed{z = \frac{H}{R} r.}$$

We first integrate with respect to r. For a fixed value of z, the radius r starts from the z-axis:

$$r = 0,$$

and ends at the surface of the cone:

$$r = \frac{R}{H} z.$$

Therefore,

$$\boxed{0 \leq r \leq \frac{R}{H} z.}$$

The variable z goes from the vertex to the base:

$$\boxed{0 \leq z \leq H.}$$

The angle θ makes one complete revolution:

$$\boxed{0 \leq \theta \leq 2\pi.}$$

Thus,

$$\boxed{0 \leq \theta \leq 2\pi, \qquad 0 \leq z \leq H, \qquad 0 \leq r \leq \frac{R}{H} z.}$$

Therefore,

$$\boxed{V = \int_0^{2\pi} \int_0^{H} \int_0^{\frac{R}{H} z} r \, dr \, dz \, d\theta.}$$

The order is

$$\boxed{dr \, dz \, d\theta}.$$

So we integrate first with respect to r, then z, and finally θ. Integration with Respect to r. We have

$$\int_0^{\frac{R}{H} z} r \, dr = \left[\frac{r^2}{2} \right]_0^{\frac{R}{H} z}.$$

Therefore,

$$\int_0^{\frac{R}{H} z} r \, dr = \frac{1}{2} \left(\frac{R}{H} z \right)^2.$$

Hence,

$$\boxed{\int_0^{\frac{R}{H}z} r\,dr = \frac{R^2z^2}{2H^2}.}$$

Thus,

$$V = \frac{R^2}{2H^2}\int_0^{2\pi}\int_0^{H} z^2\,dz\,d\theta.$$

Integration with Respect to z. Since

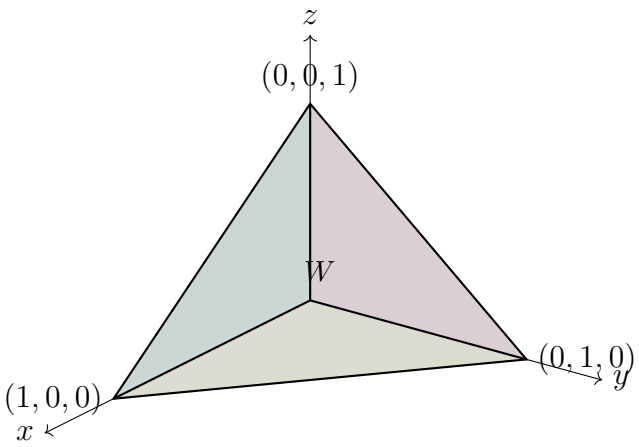
$$\int z^2\,dz = \frac{z^3}{3},$$

we obtain

$$V = \frac{R^2}{2H^2}\int_0^{2\pi}\left[\frac{z^3}{3}\right]_0^{H} d\theta = \frac{R^2}{2H^2}\int_0^{2\pi}\frac{H^3}{3}\,d\theta = \frac{\pi R^2 H}{3}.$$

---

2) Mass $\iiint_W z\,\mathrm{d}V$, W: solid tetrahedron $x, y, z \geq 0,\ x + y + z \leq 1$.



Solution. Slice by $z \in [0,1]$: for fixed z, (x,y) ranges over the triangle $x, y \geq 0,\ x + y \leq 1 - z$, whose area is $\frac{1}{2}(1-z)^2$. Hence

$$\iiint_W z\,\mathrm{d}V = \int_0^1 z\cdot\frac{1}{2}(1-z)^2\,\mathrm{d}z = \frac{1}{2}\int_0^1 \left(z - 2z^2 + z^3\right)\mathrm{d}z = \frac{1}{2}\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) = \frac{1}{2}\cdot\frac{1}{12} = \frac{1}{24}.$$

The volume of a Tetrahedron

Consider the tetrahedron bounded by

$$x = 0, \qquad y = 0, \qquad z = 0,$$

and the plane

$$x + y + z = 1.$$

We want to calculate its volume. The coordinate planes $x = 0$, $y = 0$, and $z = 0$ meet at

$$O = (0, 0, 0).$$

The plane

$$x + y + z = 1$$

meets the three coordinate axes at:

$$A = (1, 0, 0), \qquad B = (0, 1, 0), \qquad C = (0, 0, 1).$$

Therefore, the tetrahedron has four vertices:

$$O = (0, 0, 0), \quad A = (1, 0, 0), \quad B = (0, 1, 0), \quad C = (0, 0, 1).$$

The tetrahedron has a triangular base in the plane $x + y + z = 1$, and the vertex

$$O = (0, 0, 0)$$

is below this triangular base. The tetrahedron can also be described by

$$x \geq 0, \qquad y \geq 0, \qquad z \geq 0, \qquad x + y + z \leq 1.$$

For a fixed x and y, we have

$$0 \leq z \leq 1 - x - y.$$

Also,

$$0 \leq x \leq 1, \qquad 0 \leq y \leq 1 - x.$$

Therefore,

$$V = \iiint_T 1 \, dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz \, dy \, dx.$$

First, integrate with respect to z:

$$V = \int_0^1 \int_0^{1-x} (1 - x - y) \, dy \, dx.$$

Then,

$$V = \int_0^1 \frac{(1 - x)^2}{2} \, dx.$$

Finally,

$$V = \frac{1}{2}\int_0^1 (1-x)^2\,dx = \frac{1}{2}\cdot\frac{1}{3}.$$

Hence,

$$\boxed{V = \frac{1}{6}}.$$

Generalization

More generally, consider the tetrahedron bounded by

$$x = 0, \qquad y = 0, \qquad z = 0,$$

and

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1,$$

where $a, b, c > 0$.
The three intercepts are

$$A = (a, 0, 0), \qquad B = (0, b, 0), \qquad C = (0, 0, c).$$

The tetrahedron is a fraction $1/6$ of the rectangular box with side lengths a, b, c. Therefore,

$$\boxed{V = \frac{abc}{6}}.$$

For $a = b = c = 1$, this gives

$$V = \frac{1\cdot 1\cdot 1}{6} = \boxed{\frac{1}{6}}.$$

2.4.4 Exercises from Previous Mathematics 3 Examinations

Exercise 11. (2016). Compute

$$I = \iint_D \sqrt{1 + x^2 + y^2}\,dx\,dy,$$

where $D = \{(x, y) : x \geq 0,\ x \leq y,\ 1 \leq x^2 + y^2 \leq 4\}$.

Solution. In polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$. The radial bounds are $1 \leq r \leq 2$. The conditions $x \geq 0$ and $x \leq y$ give $\pi/4 \leq \theta \leq \pi/2$. Since $dx\,dy = r\,dr\,d\theta$,

$$I = \int_{\pi/4}^{\pi/2}\int_1^2 r\sqrt{1 + r^2}\,dr\,d\theta.$$

With $u = 1 + r^2$, $du = 2r\,dr$,

$$\int_1^2 r\sqrt{1 + r^2}\,dr = \frac{1}{2}\int_2^5 u^{1/2}\,du = \frac{1}{3}\left(5^{3/2} - 2^{3/2}\right) = \frac{5\sqrt{5} - 2\sqrt{2}}{3}.$$

The angular interval has length $\pi/4$. Therefore

$$\boxed{I = \frac{\pi(5\sqrt{5} - 2\sqrt{2})}{12}}.$$

Exercise 12. (2025). Using polar coordinates, compute $\iint_D e^{-(x^2+y^2)}\,dx\,dy$, where $D = \{(x, y) \in \mathbb{R}^2 : x \geq 0,\ y \geq 0\}$.

Solution. In polar coordinates the domain is $0 \leq \theta \leq \pi/2$ and $0 \leq r < \infty$, and $dx\,dy = r\,dr\,d\theta$. Therefore

$$\iint_D e^{-(x^2+y^2)}dxdy = \int_0^{\pi/2} \int_0^{\infty} e^{-r^2} r\,dr\,d\theta.$$

Using $u = r^2$, $du = 2r\,dr$, the inner integral equals $1/2$. Hence the answer is $\boxed{\pi/4}$.

Exercise 13. (2026). Let D be the disk of radius 2 centered at the origin. Using polar coordinates, compute $\iint_D (x^2 + y^2)\,dx\,dy$.

Solution. The disk is described by $0 \leq r \leq 2$, $0 \leq \theta \leq 2\pi$. Since $x^2 + y^2 = r^2$ and $dx\,dy = r\,dr\,d\theta$,

$$\iint_D (x^2 + y^2)dxdy = \int_0^{2\pi} \int_0^2 r^3\,dr\,d\theta = 2\pi \left[\frac{r^4}{4}\right]_0^2 = \boxed{8\pi}.$$

Exercise 14. (2024). Using polar coordinates, calculate $I = \int_0^{+\infty} \int_0^{+\infty} e^{-(x^2+y^2)}\,dx\,dy$.

Solution. Set $x = r\cos\theta$, $y = r\sin\theta$, $x^2 + y^2 = r^2$, and $dx\,dy = r\,dr\,d\theta$. The first quadrant corresponds to $0 \leq \theta \leq \pi/2$ and $0 \leq r < \infty$. Thus

$$I = \int_0^{\pi/2} \int_0^{\infty} e^{-r^2} r\,dr\,d\theta = \frac{\pi}{2} \lim_{t\to\infty} \int_0^t e^{-r^2} r\,dr.$$

Since $\int_0^t e^{-r^2} r\,dr = \frac{1}{2}(1 - e^{-t^2})$, the radial limit is $1/2$. Therefore $I = \boxed{\pi/4}$.

Exercise 15. (2016). Calculate

$$I = \iint_D \frac{dx\,dy}{(1 + x + y)^2}, \qquad D = \{(x, y) : x \geq 0,\ y \geq 0,\ x + y \leq 1\}.$$

Solution. The triangular region is $0 \leq x \leq 1$, $0 \leq y \leq 1 - x$. Integrating first with respect to y gives

$$I = \int_0^1 \left[-\frac{1}{1 + x + y}\right]_{y=0}^{1-x} dx = \int_0^1 \left(\frac{1}{1 + x} - \frac{1}{2}\right) dx.$$

Hence

$$I = \Big[\ln(1+x) - \frac{x}{2}\Big]_0^1 = \boxed{\ln 2 - \frac{1}{2}}.$$

Exercise 16. (2023). Calculate

$$I = \iint_D 2x^2 y \, dx \, dy, \qquad D = \{(x,y) : x \geq 0,\ y \geq 0,\ x + y \leq 1\}.$$

Solution. For $0 \leq x \leq 1$, the variable y runs from 0 to $1 - x$. Thus

$$I = \int_0^1 \int_0^{1-x} 2x^2 y \, dy \, dx = \int_0^1 x^2 (1-x)^2 \, dx.$$

Expanding and integrating,

$$I = \int_0^1 (x^2 - 2x^3 + x^4) dx = \frac{1}{3} - \frac{1}{2} + \frac{1}{5} = \boxed{\frac{1}{30}}.$$

Chapter 3

Numerical Series

3.1 Definitions and Convergence

An infinite series $\sum_{n=1}^{\infty} a_n$ converges to S if its sequence of partial sums $S_k = \sum_{n=1}^{k} a_n$ satisfies $\lim_{k\to\infty} S_k = S$.

Definition – Divergent Series and Remainder

A series that does not converge is said to **diverge**. If $\sum a_n$ converges to S, the **remainder** $R_k = S - S_k = \sum_{n=k+1}^{\infty} a_n$ satisfies $R_k \to 0$.

Theorem – Necessary Condition for Convergence

If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$. Equivalently, if $a_n \not\to 0$, the series diverges (**term test** / *divergence test*).

Property – Cauchy Criterion

The series $\sum a_n$ converges if and only if for every $\varepsilon > 0$ there exists N such that for all $q > p \geq N$,

$$\left| \sum_{n=p+1}^{q} a_n \right| < \varepsilon.$$

Example 1 – The Geometric Series

For $|q| < 1$, $\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$; the series diverges for $|q| \geq 1$. For instance, $\sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n = \frac{1}{1-1/3} = \frac{3}{2}$.

Example 2 – The Harmonic Series Diverges

The series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, even though $\frac{1}{n} \to 0$: grouping terms in blocks of powers of two,

$$\sum_{n=1}^{2^k} \frac{1}{n} \geq 1 + \frac{1}{2} + \underbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}_{\geq 1/2} + \underbrace{\left(\frac{1}{5} + \cdots + \frac{1}{8}\right)}_{\geq 1/2} + \cdots \geq 1 + \frac{k}{2} \to \infty.$$

This shows the necessary condition $a_n \to 0$ is *not* sufficient for convergence.

3.2 Series with Non-Negative Terms

Proposition – Riemann Series

The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.

Theorem – Comparison Test

Let $0 \leq a_n \leq b_n$ for all $n \geq N$.

- If $\sum b_n$ converges, then $\sum a_n$ converges.
- If $\sum a_n$ diverges, then $\sum b_n$ diverges.

Theorem – Equivalence Test

If $a_n, b_n \geq 0$ and $a_n \sim b_n$ (i.e. $\lim \frac{a_n}{b_n} = L$ with $0 < L < \infty$), then $\sum a_n$ and $\sum b_n$ have the same nature.

Theorem – d'Alembert's Ratio Test

Let $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$.

- If $L < 1$, the series converges absolutely.
- If $L > 1$, the series diverges.
- If $L = 1$, the test is inconclusive.

Theorem – Cauchy's Root Test

Let $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$.

- If $L < 1$, the series converges absolutely.
- If $L > 1$, the series diverges.
- If $L = 1$, the test is inconclusive.

Theorem – Integral Test

Let $f : [1, \infty) \to \mathbb{R}_{\geq 0}$ be continuous, decreasing, with $f(n) = a_n$. Then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\int_1^{\infty} f(x)\,dx$ converges.

Example 1 – Ratio Test in Action

Study $\sum_{n=1}^{\infty} \frac{n}{2^n}$. With $a_n = n/2^n$,

$$\frac{a_{n+1}}{a_n} = \frac{n+1}{2n} \xrightarrow[n\to\infty]{} \frac{1}{2} < 1,$$

so the series converges absolutely.

Example 2 – Root Test in Action

Study $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$. Here

$$\sqrt[n]{a_n} = \left(\frac{n}{n+1}\right)^{n} = \left(1 - \frac{1}{n+1}\right)^{n} \xrightarrow[n\to\infty]{} e^{-1} < 1,$$

so the series converges.

3.3 Alternating Series and Absolute Convergence

Definition – Absolute and Conditional Convergence

The series $\sum a_n$ **converges absolutely** if $\sum |a_n|$ converges. If $\sum a_n$ converges but $\sum |a_n|$ diverges, the series is said to **converge conditionally**.

Theorem – Absolute Convergence Implies Convergence

If $\sum |a_n|$ converges, then $\sum a_n$ converges.

Theorem – Leibniz's Alternating Series Test

Let (b_n) be a decreasing sequence of positive numbers with $b_n \to 0$. Then the alternating series $\sum_{n=1}^{\infty} (-1)^n b_n$ converges. Moreover, the remainder satisfies $|R_k| \le b_{k+1}$.

Example 1 – The Alternating Harmonic Series

The series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges by the Leibniz test (since $b_n = 1/n$ decreases to 0), with sum $\ln 2$. However, $\sum \frac{1}{n}$ diverges, so the convergence is only **conditional**.

3.4 Additional Definitions and Convergence Principles

Definition – Partial Sum and Remainder

For $\sum_{n=1}^{\infty} a_n$, its Nth partial sum is

$$S_N = \sum_{n=1}^{N} a_n.$$

If the series converges to S, the remainder after N terms is

$$R_N = S - S_N = \sum_{n=N+1}^{\infty} a_n.$$

Definition – Absolute and Conditional Convergence

A series $\sum a_n$ is **absolutely convergent** if $\sum |a_n|$ converges. It is **conditionally convergent** if $\sum a_n$ converges but $\sum |a_n|$ diverges.

Theorem – Comparison Test for Positive Series

If $0 \le a_n \le b_n$ for all sufficiently large n and $\sum b_n$ converges, then $\sum a_n$ converges. If $0 \le b_n \le a_n$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

Theorem – Limit Comparison Test

Let $a_n, b_n > 0$ for all sufficiently large n. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L, \qquad 0 < L < \infty,$$

then $\sum a_n$ and $\sum b_n$ have the same nature.

Theorem – Ratio Test

For a series with $a_n \neq 0$, let

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

If $L < 1$, the series converges absolutely; if $L > 1$, it diverges; when $L = 1$, the test is inconclusive.

Theorem – Root Test

Let

$$L = \limsup_{n\to\infty} \sqrt[n]{|a_n|}.$$

If $L < 1$, $\sum a_n$ converges absolutely; if $L > 1$, it diverges. For $L = 1$ the test gives no

conclusion.

3.5 Alternating Series and Estimates

Theorem – Leibniz Alternating-Series Test

Suppose $b_n \geq 0$, $b_{n+1} \leq b_n$, and $b_n \to 0$. Then

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n$$

converges. Moreover, its remainder satisfies

$$|R_N| \leq b_{N+1}.$$

Example 1 – Alternating-Series Error Estimate

To approximate

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n},$$

using N terms, the error is at most $1/(N+1)$. Thus $N = 999$ guarantees an error smaller than 0.001.

3.6 Integral Test

Theorem – Integral Test

Let $f : [1, \infty) \to (0, \infty)$ be continuous and decreasing, with $a_n = f(n)$. Then

$$\sum_{n=1}^{\infty} a_n$$

converges if and only if

$$\int_1^{\infty} f(x)\, dx$$

converges.

Example 1 – A Logarithmic Series

Study

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}.$$

For $f(x) = 1/[x(\ln x)^p]$, the substitution $u = \ln x$ gives

$$\int_2^\infty \frac{dx}{x(\ln x)^p} = \int_{\ln 2}^\infty u^{-p}\,du.$$

Hence the series converges exactly when $p > 1$.

3.7 Powerful Comparison Techniques

Proposition – Asymptotic Equivalence

If $a_n \sim b_n$, meaning

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 1,$$

and both sequences are eventually positive, then $\sum a_n$ and $\sum b_n$ have the same convergence behavior.

Example 1 – Equivalence with a Polynomial

Study

$$\sum_{n=1}^{\infty} \frac{3n^2+1}{n^5+2n}.$$

Dividing numerator and denominator by n^5 shows

$$\frac{3n^2+1}{n^5+2n} \sim \frac{3}{n^3}.$$

Since $\sum 1/n^3$ converges, the given series converges.

3.8 Series with Special Structure

Definition – Telescoping Series

A series is **telescoping** when cancellation between consecutive terms reduces its partial sums to a small number of boundary terms. A common form is

$$a_n = b_n - b_{n+1}.$$

Then

$$\sum_{n=1}^{N} b_n - b_{n+1} = b_1 - b_{N+1}.$$

Example 1 – A More General Telescoping Series

Compute

$$\sum_{n=1}^{\infty} \frac{1}{n(n+2)}.$$

Since

$$\frac{1}{n(n+2)} = \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+2}\right),$$

the partial sums reduce to

$$S_N = \frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{N+1} - \frac{1}{N+2}\right).$$

Thus

$$\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \frac{3}{4}.$$

Definition – Cauchy Product

For two absolutely convergent series

$$\sum_{n=0}^{\infty} a_n, \qquad \sum_{n=0}^{\infty} b_n,$$

their product can be written as the Cauchy product

$$\sum_{n=0}^{\infty} c_n, \qquad c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

Under absolute convergence, the product series converges to the product of the two sums.

Example 2 – Product of Geometric Series

For $|x| < 1$,

$$\frac{1}{(1-x)^2} = \left(\sum_{n=0}^{\infty} x^n\right)^2.$$

The coefficient of x^n is $n+1$, so

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1)x^n.$$

This provides a useful connection between numerical series and power series.

3.9 Chapter Review

A reliable strategy for a numerical series is to inspect its general term first. If the term does not tend to zero, the series diverges immediately. Otherwise choose a test suited to

the structure: comparison or limit comparison for positive terms, the integral test for monotone functions, ratio or root tests for factorials and powers, and the alternating test when signs alternate. Absolute convergence is stronger than ordinary convergence and is often the preferred conclusion.

3.10 Worked Exercises

Exercise 1. Nature of a Series by the Equivalence Test. Study the convergence of $\sum_{n=1}^{\infty} \frac{2n^2+1}{n^4-n+3}$.

Solution. As $n \to \infty$, $a_n = \frac{2n^2+1}{n^4-n+3} \sim \frac{2n^2}{n^4} = \frac{2}{n^2}$. Since $\sum \frac{2}{n^2}$ is (up to a constant) a convergent Riemann series ($p = 2 > 1$), the equivalence test shows $\sum a_n$ converges.

Exercise 2. Ratio Test with Factorials. Study the convergence of $\sum_{n=1}^{\infty} \frac{n!}{n^n}$.

Solution. Let $a_n = n!/n^n$. Then

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \frac{n^n}{(n+1)^n} = \left(\frac{n}{n+1}\right)^n = \left(1+\frac{1}{n}\right)^{-n} \xrightarrow[n\to\infty]{} e^{-1} < 1.$$

By the ratio test, the series converges.

Exercise 3. Absolute vs Conditional Convergence. Determine whether $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges absolutely, conditionally, or diverges.

Solution. Taking absolute values gives $\sum \frac{1}{\sqrt{n}}$, a Riemann series with $p = 1/2 < 1$, which diverges — so there is no absolute convergence. However $b_n = 1/\sqrt{n}$ is decreasing to 0, so by the Leibniz test the alternating series converges. Hence the series converges **conditionally**.

Exercise 4. Sum of a Telescoping Series. Compute $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

Solution. Using partial fractions $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$, the partial sum telescopes:

$$S_k = \sum_{n=1}^{k} \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{k+1} \xrightarrow[k\to\infty]{} 1.$$

So the series converges and its sum is 1.

3.10.1 Additional Worked Exercises

Exercise 5. Integral Test. Study the convergence of

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}.$$

Solution. The associated integral is

$$\int_2^{\infty} \frac{dx}{x(\ln x)^2}.$$

Set $u = \ln x$, $du = dx/x$. Then

$$\int_{\ln 2}^{\infty} u^{-2}\,du = \left[-\frac{1}{u}\right]_{\ln 2}^{\infty} = \frac{1}{\ln 2} < \infty.$$

Therefore the series converges.

Exercise 6. Ratio Test. Study

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n}.$$

Solution. Let $a_n = n^2/2^n$. Then

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^2}{2n^2} \to \frac{1}{2} < 1.$$

Hence the series converges absolutely.

Exercise 7. Root Test. Study

$$\sum_{n=1}^{\infty} \left(\frac{2n+1}{3n+4}\right)^n.$$

Solution. The nth root of the absolute value is

$$\frac{2n+1}{3n+4} \to \frac{2}{3} < 1.$$

The root test therefore proves convergence.

Exercise 8. Conditional Convergence. Determine the nature of

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\ln n}{n}.$$

Solution. The terms $b_n = \ln n/n$ do not decrease to zero eventually; in fact, they increase up to e and then decrease, but importantly $b_n \to 0$. For n sufficiently large, b_n is decreasing. Thus the alternating-series test gives convergence. However

$$\sum \frac{\ln n}{n}$$

diverges by the integral test because

$$\int_2^{\infty} \frac{\ln x}{x}\,dx = \frac{1}{2}(\ln x)^2\Big|_2^{\infty} = \infty.$$

Therefore the series is conditionally convergent.

Exercise 9. Telescoping with a Shift. Compute

$$\sum_{n=1}^{\infty} \frac{1}{n(n+3)}.$$

Solution. We have

$$\frac{1}{n(n+3)} = \frac{1}{3}\left(\frac{1}{n} - \frac{1}{n+3}\right).$$

The first three terms survive after cancellation, so

$$S = \frac{1}{3}\left(1 + \frac{1}{2} + \frac{1}{3}\right) = \frac{11}{18}.$$

Exercise 10. A Logarithmic Estimate. Show that

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

diverges.

Solution. The function $f(x) = 1/(x \ln x)$ is positive and decreasing for $x > e$. By the integral test,

$$\int_2^{\infty} \frac{dx}{x \ln x} = [\ln(\ln x)]_2^{\infty} = \infty.$$

Hence the series diverges.

Exercise 11. Review Exercise. Determine the nature of $\sum_{n=1}^{\infty} (-1)^{n-1}/n^2$.

Solution. Since $\sum 1/n^2$ converges, the given series converges absolutely.

Exercise 12. Comparison. Study $\sum_{n=1}^{\infty} 1/(n^2+3)$.

Solution. Since $0 < 1/(n^2+3) \leq 1/n^2$ and the p-series with $p = 2$ converges, the given series converges.

3.10.2 Exercises from Previous Mathematics 3 Examinations

Exercise 13. (2014). Study the convergence of

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + |\cos n|}.$$

Solution. Since $0 \leq |\cos n| \leq 1$, we have

$$0 < \frac{1}{n^2 + |\cos n|} \leq \frac{1}{n^2}.$$

The p-series $\sum 1/n^2$ converges because $p = 2 > 1$. The comparison test therefore proves that the given series converges.

Exercise 14. (2014). For a real parameter $a \geq 0$, determine the convergence of

$$\sum_{n=0}^{\infty} \ln(1 + a^n).$$

Solution. If $a > 1$, then $a^n \to +\infty$ and $\ln(1+a^n) \not\to 0$, so the series diverges by the necessary condition for convergence. If $a = 1$, every term equals $\ln 2 > 0$, so it diverges. If $a = 0$, all terms with $n \geq 1$ are zero (and the $n = 0$ term is $\ln 2$ under the convention $0^0 = 1$; if the series is instead indexed from $n = 1$, its sum is zero). Finally, for $0 < a < 1$, we have $\ln(1+a^n) \sim a^n$ as $n \to \infty$. The geometric series $\sum a^n$ converges, so limit comparison proves convergence. Thus the series converges for $\boxed{0 \leq a < 1}$ and diverges for $\boxed{a \geq 1}$.

Exercise 15. (2026). Study the convergence of the following series:

$$\text{(a)} \sum_{n=1}^{\infty} \frac{n^2}{3^n}, \qquad \text{(b)} \sum_{n=1}^{\infty} \frac{(2n+1)^n}{(3n-1)^n}, \qquad \text{(c)} \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}.$$

Solution. **(a)** $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{3^n}$. **Ratio test.** Set $a_n = n^2/3^n$. The ratio test says that if $a_n > 0$ and $L = \lim_{n\to\infty} a_{n+1}/a_n$ exists, then the series converges when $L < 1$ and diverges when $L > 1$; if $L = 1$, the test is inconclusive. Here

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^2}{3^{n+1}} \frac{3^n}{n^2} = \frac{1}{3}\left(1 + \frac{1}{n}\right)^2,$$

so

$$L = \lim_{n\to\infty} \frac{1}{3}\left(1 + \frac{1}{n}\right)^2 = \frac{1}{3} < 1.$$

Thus the series converges. Since its terms are non-negative, this is also absolute convergence.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(2n+1)^n}{(3n-1)^n}$. **Root test.** Let $a_n = ((2n+1)/(3n-1))^n$. The root test uses $L = \lim \sqrt[n]{|a_n|}$: if $L < 1$, the series converges; if $L > 1$, it diverges; if $L = 1$, no conclusion follows. In this exercise,

$$\sqrt[n]{a_n} = \frac{2n+1}{3n-1} = \frac{2 + 1/n}{3 - 1/n} \longrightarrow \frac{2}{3} < 1.$$

Therefore the root test proves that the series converges (absolutely).

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$. **Alternating-series test.** Write the series as $\sum (-1)^n b_n$, where $b_n = 1/\sqrt{n}$. The alternating-series test states that $\sum (-1)^n b_n$ converges if $b_n \geq 0$, b_n is decreasing from some index onward, and $b_n \to 0$. Here b_n is positive, decreases as n increases, and

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0.$$

So the alternating series converges.

To decide whether it converges absolutely, examine $\sum |(-1)^n/\sqrt{n}| = \sum 1/n^{1/2}$. The p-series rule says $\sum 1/n^p$ converges exactly when $p > 1$; here $p = 1/2 \leq 1$, so this

absolute-value series diverges. Hence the original series is **conditionally convergent**, not absolutely convergent.

Exercise 16. (2025). Study convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(1/n)}{n}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$.

Solution. **(a) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(1/n)}{n}$. Limit comparison with the harmonic series.** For positive-term series, the limit comparison test states that if $a_n, b_n > 0$ eventually and $\lim a_n/b_n = L$ with $0 < L < \infty$, then $\sum a_n$ and $\sum b_n$ have the same nature. Take

$$a_n = \frac{\cos(1/n)}{n}, \qquad b_n = \frac{1}{n}.$$

For all sufficiently large n, $\cos(1/n) > 0$, and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \cos(1/n) = 1.$$

The limit is strictly between zero and infinity, so the two series have the same nature. The harmonic series $\sum 1/n$ diverges (the p-series boundary case $p = 1$), hence $\sum \cos(1/n)/n$ diverges.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$. Absolute convergence and the p-series rule. Consider the absolute-value series:

$$\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n^3}\right| = \sum_{n=1}^{\infty} \frac{1}{n^3}.$$

The p-series $\sum 1/n^p$ converges if and only if $p > 1$. Here $p = 3 > 1$, so the absolute-value series converges. Therefore the given alternating series converges **absolutely** (which implies ordinary convergence).

Exercise 17. (2025). Study the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^2}$.

Solution. For every integer $n \geq 1$, $0 \leq \sin^2 n \leq 1$. Dividing by $n^2 > 0$ gives

$$0 \leq \frac{\sin^2 n}{n^2} \leq \frac{1}{n^2}.$$

The comparison test for series states that if $0 \leq a_n \leq b_n$ for every sufficiently large n and $\sum b_n$ converges, then $\sum a_n$ also converges. Here the comparison series $\sum 1/n^2$ is a p-series with $p = 2 > 1$, so it converges. All the hypotheses of the comparison test are satisfied, and therefore

$$\boxed{\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^2} \text{ converges.}}$$

In fact, the same estimate shows absolute convergence, since the terms of the given series are already non-negative. The comparison test establishes convergence; it does not determine the exact sum.

Exercise 18. (2024). Study the nature of $\sum_{n=0}^{\infty}(1/2)^{-n}$ and $\sum_{n=1}^{\infty}\left(\int_0^{\pi} x\sin(nx)\,dx\right)$. If $\sum_{n=1}^{\infty} u_n$ converges with $u_n \geq 0$, also study $\sum\sqrt{1+u_n}$ and $\sum\ln(1+u_n)$.

Solution. For the first series, $(1/2)^{-n} = 2^n$ does not tend to zero. The necessary condition for convergence fails; also the root and ratio limits are both $2 > 1$. It is a geometric series of ratio 2, so it diverges.

For $u_n = \int_0^{\pi} x\sin(nx)\,dx$, integration by parts gives

$$u_n = \left[-\frac{x\cos(nx)}{n}\right]_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos(nx)\,dx = \frac{\pi(-1)^{n+1}}{n}.$$

The resulting series is $\pi\sum_{n=1}^{\infty}(-1)^{n+1}/n$, which converges by the Leibniz test because $1/n$ decreases to zero. Abel's test also applies: the coefficients $1/n$ decrease to zero and the partial sums of $\pi(-1)^{n+1}$ are bounded.

Since $\sum u_n$ converges with non-negative terms, $u_n \to 0$. Hence $\sqrt{1+u_n} \to 1 \neq 0$, so $\sum\sqrt{1+u_n}$ diverges by the necessary condition. Finally, $\ln(1+u_n) \sim u_n$ as $u_n \to 0$, so the equivalence test shows that $\sum\ln(1+u_n)$ converges.

Exercise 19. (2023). Study the convergence of

$$\sum_{n=1}^{\infty}\frac{1}{n+|\cos n|}.$$

Solution. Since $|\cos n| \leq 1$, we have $n+|\cos n| \leq n+1$, and hence

$$\frac{1}{n+|\cos n|} \geq \frac{1}{n+1}.$$

The harmonic series $\sum_{n=1}^{\infty} 1/(n+1)$ diverges. By comparison, the given positive-term series diverges.

Exercise 20. (2014). Study the convergence and absolute convergence of

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n^2+n}}.$$

Solution. Let $b_n = 1/\sqrt{n^2+n}$. The sequence b_n is positive, decreases to zero, and therefore the alternating-series (Leibniz) test proves convergence. For absolute convergence, note that

$$\frac{1}{\sqrt{n^2+n}} = \frac{1}{n\sqrt{1+1/n}} \sim \frac{1}{n}.$$

By limit comparison with the harmonic series, $\sum b_n$ diverges. The original series is therefore conditionally convergent, not absolutely convergent.

Exercise 21. (2023). Study the convergence of

$$\sum_{n=1}^{\infty}\ln\left(1+\frac{1}{n^2}\right).$$

Solution. For every $t \geq 0$, $0 \leq \ln(1+t) \leq t$. Consequently,

$$0 \leq \ln\left(1 + \frac{1}{n^2}\right) \leq \frac{1}{n^2}.$$

Since the p-series $\sum 1/n^2$ converges, the comparison test proves that the given series converges.

Chapter 4

Sequences, Series of Functions, and Power Series

4.1 Sequences and Series of Functions

Definition – Pointwise Convergence

A sequence of functions $f_n : I \to \mathbb{R}$ **converges pointwise** to f on I if for every $x \in I$, $\lim_{n\to\infty} f_n(x) = f(x)$.

Definition – Pointwise vs Uniform Convergence

A sequence $f_n(x)$ converges **uniformly** to $f(x)$ on I if:

$$\lim_{n\to\infty} \sup_{x\in I} |f_n(x) - f(x)| = 0.$$

Property – Uniform Convergence Implies Pointwise Convergence

If $f_n \to f$ uniformly on I, then $f_n \to f$ pointwise on I. The converse is false in general.

Example 1 – Pointwise but not Uniform Convergence

Let $f_n(x) = x^n$ on $[0,1]$. Pointwise, $f_n(x) \to f(x)$ where $f(x) = 0$ for $x \in [0,1)$ and $f(1) = 1$. But

$$\sup_{x\in[0,1]} |f_n(x) - f(x)| = \sup_{x\in[0,1)} x^n = 1 \not\to 0,$$

so the convergence is *not* uniform on $[0,1]$ (it is, however, uniform on any $[0,a]$ with $a < 1$).

Theorem – Continuity of the Uniform Limit

If each f_n is continuous on I and $f_n \to f$ uniformly on I, then f is continuous on I.

Proposition – Weierstrass M-Test

Let $\sum u_n(x)$ be a series of functions on I. If there exist constants $M_n \geq 0$ with $|u_n(x)| \leq M_n$ for all $x \in I$ and $\sum M_n$ converges, then $\sum u_n(x)$ converges uniformly (and absolutely)

on I.

Theorem – Term-by-Term Integration and Differentiation

Suppose $f_n \to f$ uniformly on $[a, b]$ and each f_n is continuous. Then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \int_a^b f_n(x)\,dx.$$

If moreover each f_n is C^1, $f_n \to f$ pointwise, and $f_n' \to g$ uniformly, then f is differentiable with $f' = g$.

Example 2 – Applying the M-Test

The series $\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2}$ converges uniformly on $\mathbb{R}$, since $\left|\frac{\cos(nx)}{n^2}\right| \leq \frac{1}{n^2}$ and $\sum \frac{1}{n^2}$ converges (Riemann series, $p = 2$).

4.2 Power Series

Definition – Power Series & Radius of Convergence

A power series centered at x_0 has the form $\sum_{n=0}^{\infty} c_n(x - x_0)^n$. Its radius of convergence R is given by Cauchy-Hadamard formula:

$$\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|c_n|}.$$

Property – Interval of Convergence

A power series $\sum c_n(x - x_0)^n$ with radius of convergence R converges absolutely for $|x - x_0| < R$ and diverges for $|x - x_0| > R$. At $|x - x_0| = R$, convergence must be checked separately (it may converge, diverge, or converge only at one endpoint).

Proposition – Ratio Formula for the Radius

If $\lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| = \ell$ exists, then $R = 1/\ell$ (with $R = \infty$ if $\ell = 0$, and $R = 0$ if $\ell = \infty$).

Theorem – Term-by-Term Differentiation and Integration of Power Series

Within the open interval of convergence $|x - x_0| < R$, a power series $f(x) = \sum_{n=0}^{\infty} c_n(x - x_0)^n$ can be differentiated and integrated term by term, and the resulting series have the

same radius of convergence R:

$$f'(x) = \sum_{n=1}^{\infty} n c_n (x - x_0)^{n-1}, \qquad \int f(x)\,dx = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x - x_0)^{n+1}.$$

Theorem – Abel's Theorem

If a power series $\sum c_n(x - x_0)^n$ with radius R converges at an endpoint $x = x_0 + R$, then its sum function is continuous at that point from the interior, i.e.

$$\lim_{x \to (x_0+R)^-} f(x) = \sum_{n=0}^{\infty} c_n R^n.$$

Example 1 – Radius of Convergence

For $\sum_{n=0}^{\infty} \frac{x^n}{n!}$, we have $c_n = 1/n!$, so

$$\left|\frac{c_{n+1}}{c_n}\right| = \frac{n!}{(n+1)!} = \frac{1}{n+1} \to 0,$$

hence $R = \infty$: the series converges for every $x \in \mathbb{R}$ (this is precisely the Taylor series of e^x).

Example 2 – Standard Taylor / Maclaurin Series

The following power series are fundamental and valid on the indicated intervals:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}\ (R = \infty), \qquad \sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}\ (R = \infty),$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}\ (R = \infty), \qquad \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n\ (R = 1),$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n\ (R = 1).$$

4.3 Further Definitions for Functional Sequences

Definition – Pointwise Convergence

A sequence of functions (f_n) on D converges **pointwise** to f if for every fixed $x \in D$,

$$\lim_{n\to\infty} f_n(x) = f(x).$$

The pointwise rate of convergence may depend strongly on x.

Definition – Uniform Convergence

The sequence f_n converges **uniformly** to f on D if

$$\forall \varepsilon > 0 \ \exists N \text{ such that } n \geq N \implies |f_n(x) - f(x)| < \varepsilon$$

for every $x \in D$. Equivalently,

$$\sup_{x\in D} |f_n(x) - f(x)| \to 0.$$

Theorem – Uniform Limit of Continuous Functions

If each f_n is continuous on D and $f_n \to f$ uniformly on D, then f is continuous on D.

Theorem – Uniform Convergence and Integration

If f_n are Riemann integrable on $[a, b]$ and $f_n \to f$ uniformly, then f is Riemann integrable and

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx.$$

Theorem – Uniform Convergence and Differentiation

Suppose f_n are differentiable on $[a, b]$, the derivatives f_n' converge uniformly to g, and $f_n(x_0)$ converges for at least one $x_0 \in [a, b]$. Then f_n converges uniformly to a differentiable function f and

$$f' = g.$$

4.4 Weierstrass-Type Criteria

Theorem – Weierstrass M-Test

Let $\sum f_n(x)$ be a series of functions on D. If there exist constants $M_n \geq 0$ such that

$$|f_n(x)| \leq M_n \quad \text{for all } x \in D,$$

and $\sum M_n$ converges, then $\sum f_n$ converges uniformly and absolutely on D.

Example 1 – Uniform Convergence by the M-Test

On $[-1,1]$, consider

$$\sum_{n=1}^{\infty} \frac{x^n}{n^2}.$$

Since $|x^n|/n^2 \le 1/n^2$ and $\sum 1/n^2$ converges, the M-test gives uniform and absolute convergence.

4.5 Power Series

Definition – Power Series

A **power series centered at** x_0 is a series of the form

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

There exists a number $R \in [0, \infty]$, called the **radius of convergence**, such that the series converges absolutely for $|x - x_0| < R$ and diverges for $|x - x_0| > R$.

Theorem – Cauchy–Hadamard Formula

For

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

the radius of convergence is

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}},$$

with the usual conventions $1/0 = \infty$ and $1/\infty = 0$.

Theorem – Term-by-Term Differentiation and Integration

Inside its interval of convergence,

$$\frac{d}{dx} \sum_{n=0}^{\infty} a_n (x - x_0)^n = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1},$$

and

$$\int \sum_{n=0}^{\infty} a_n (x - x_0)^n \, dx = C + \sum_{n=0}^{\infty} \frac{a_n}{n+1} (x - x_0)^{n+1}.$$

The differentiated and integrated series have the same radius of convergence as the original series.

Example 1 – Taylor Expansion of $\arctan x$

Since

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}, \qquad |x| < 1,$$

integration from 0 to x gives

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \qquad |x| < 1.$$

At $x = 1$, this produces the classical identity

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.$$

4.6 Taylor Polynomials and Approximation

Definition – Taylor Polynomial

If f has derivatives up to order n at x_0, its Taylor polynomial of degree n is

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k.$$

It approximates f near x_0.

Theorem – Taylor's Theorem with Remainder

If f has $n+1$ derivatives on an interval containing x_0 and x, then

$$f(x) = T_n(x) + R_n(x),$$

where, for some ξ between x_0 and x,

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1}.$$

This formula provides an explicit error estimate when the derivative is bounded.

Example 1 – Approximating e^x

At $x_0 = 0$,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$$

Using the cubic polynomial at $x = 0.1$ gives

$$e^{0.1} \approx 1 + 0.1 + \frac{0.01}{2} + \frac{0.001}{6} = 1.105166\ldots.$$

The exact value is approximately 1.105170, illustrating the rapid convergence near the center.

Property – Uniqueness of Power-Series Representation

If two power series centered at the same point represent the same function on an interval with an accumulation point inside their common interval of convergence, then their coefficients are equal term by term.

4.7 Chapter Review

The distinction between pointwise and uniform convergence is fundamental. Uniform convergence allows important operations such as passage to the limit under an integral under appropriate hypotheses. For a series of functions, the Weierstrass M-test is a practical sufficient condition. For power series, determine the radius first and then examine the endpoints separately.

4.8 Worked Exercises

Exercise 1. Radius of Convergence via the Root Test. Find the radius of convergence of $\sum_{n=1}^{\infty} \frac{n}{3^n} x^n$.

Solution. Here $c_n = n/3^n$, and $\sqrt[n]{|c_n|} = \frac{\sqrt[n]{n}}{3} \to \frac{1}{3}$ since $\sqrt[n]{n} \to 1$. By Cauchy-Hadamard, $R = 3$.

Exercise 2. Interval of Convergence Including Endpoints. Find the exact interval of convergence of $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} x^n$.

Solution. With $c_n = (-1)^n/n$, $\left|\frac{c_{n+1}}{c_n}\right| = \frac{n}{n+1} \to 1$, so $R = 1$. At $x = 1$: $\sum \frac{(-1)^n}{n}$ converges (Leibniz). At $x = -1$: $\sum \frac{(-1)^n(-1)^n}{n} = \sum \frac{1}{n}$ diverges. Hence the interval of convergence is $(-1, 1]$.

Exercise 3. Summing a Power Series. Find the sum $f(x) = \sum_{n=1}^{\infty} n x^{n-1}$ for $|x| < 1$.

Solution. Since $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $|x| < 1$, differentiate term by term (valid inside the radius of convergence):

$$f(x) = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}.$$

Exercise 4. Uniform Convergence Check. Show that $f_n(x) = \frac{x}{1+nx^2}$ converges uniformly to 0 on $\mathbb{R}$.

Solution. By AM-GM, $1 + nx^2 \geq 2\sqrt{n}|x|$ for $x \neq 0$, so $|f_n(x)| = \frac{|x|}{1+nx^2} \leq \frac{|x|}{2\sqrt{n}|x|} = \frac{1}{2\sqrt{n}}$ for $x \neq 0$ (and $f_n(0) = 0$). Hence $\sup_x |f_n(x)| \leq \frac{1}{2\sqrt{n}} \to 0$, so the convergence is uniform on $\mathbb{R}$.

4.8.1 Additional Worked Exercises

Exercise 5. Pointwise but Not Uniform. Consider $f_n(x) = x^n$ on $[0, 1]$.

Solution. For $0 \leq x < 1$, $x^n \to 0$, while $f_n(1) = 1$. Thus the pointwise limit is

$$f(x) = \begin{cases} 0, & 0 \leq x < 1, \\ 1, & x = 1. \end{cases}$$

The convergence is not uniform because

$$\sup_{[0,1]} |f_n - f| = 1$$

for every n.

Exercise 6. Uniform Convergence by the M-Test. Show that

$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2}$$

converges uniformly on $\mathbb{R}$.

Solution. For every x,

$$\left| \frac{\sin(nx)}{n^2} \right| \leq \frac{1}{n^2}.$$

Since $\sum 1/n^2$ converges, the Weierstrass M-test proves uniform convergence on $\mathbb{R}$.

Exercise 7. Radius and Interval of Convergence. Find the interval of convergence of

$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{n}.$$

Solution. The ratio test gives $|x - 2| < 1$, so the radius is $R = 1$. At $x = 1$ we obtain

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n},$$

which converges. At $x = 3$ we obtain the harmonic series, which diverges. Hence the interval is

$$[1, 3).$$

Exercise 8. Term-by-Term Integration. For $|x| < 1$, integrate

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

from 0 to x.

Solution. Uniform convergence holds on every closed interval $[-r, r]$ with $r < 1$, so term-by-term integration is valid:

$$\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = \int_0^x \frac{dt}{1-t} = -\ln(1-x).$$

Thus

$$-\ln(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n}.$$

Exercise 9. Taylor Polynomial. Find the Taylor polynomial of degree 3 for $\ln(1+x)$ about $x = 0$.

Solution. The derivatives give

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots.$$

Thus

$$T_3(x) = x - \frac{x^2}{2} + \frac{x^3}{3}.$$

Exercise 10. Approximation Error. Use $T_2(x) = 1 + x + x^2/2$ to approximate $e^{0.2}$.

Solution.

$$T_2(0.2) = 1 + 0.2 + \frac{0.2^2}{2} = 1.22.$$

Taylor's theorem gives

$$|R_2(0.2)| \le \frac{e^{0.2}(0.2)^3}{6},$$

so the approximation error is less than approximately 0.00148.

Exercise 11. Review Exercise. Find the radius of convergence of $\sum_{n=0}^{\infty} x^n/2^n$.

Solution. The geometric series converges when $|x/2| < 1$, hence $|x| < 2$. Therefore $R = 2$.

Exercise 12. Power Series. Find the radius of convergence of $\sum_{n=1}^{\infty} n(x-1)^n/4^n$.

Solution. The ratio test gives

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n+1}{n}\frac{|x-1|}{4} \to \frac{|x-1|}{4}.$$

Thus convergence occurs for $|x-1| < 4$, so $R = 4$.

4.8.2 Exercises from Previous Mathematics 3 Examinations

Exercise 13. (2014). For $x \in [0, +\infty)$, let $f_n(x) = e^{-nx}\sin(nx)$, $n \ge 1$. Study pointwise convergence on $[0, +\infty)$, uniform convergence on $[0, +\infty)$ and on $[a, +\infty)$ for $a > 0$. Also calculate $\lim_{n\to\infty} \int_a^{+\infty} e^{-nx}\sin(nx)\,dx$ for $a > 0$.

Solution. For $x = 0$, $f_n(0) = 0$. For each fixed $x > 0$, $|f_n(x)| \le e^{-nx} \to 0$. Thus $f_n \to 0$ pointwise on $[0, +\infty)$.
On the full half-line, the convergence is not uniform. Indeed, taking $x = 1/n$ gives

$$|f_n(1/n)| = e^{-1}|\sin 1|,$$

which is a positive constant independent of n. Therefore $\sup_{x\ge 0}|f_n(x)|$ cannot tend to zero.
For $x \ge a > 0$,

$$|f_n(x)| \le e^{-nx} \le e^{-na} \longrightarrow 0.$$

Hence convergence is uniform on $[a, +\infty)$. To evaluate the integral, use

$$\int e^{-nx}\sin(nx)\,dx = -\frac{e^{-nx}}{2n}\big(\sin(nx) + \cos(nx)\big).$$

It follows that

$$\int_a^{+\infty} e^{-nx}\sin(nx)\,dx = \frac{e^{-na}}{2n}\big(\sin(na) + \cos(na)\big),$$

which tends to 0 as $n \to \infty$. Thus the requested limit is $\boxed{0}$.

Exercise 14. (2026). Let $f_n(x) = x^n(1 - x)$ for $x \in [0, 1]$. Find the pointwise limit, study uniform convergence on $[0, 1]$, and determine whether the limit may be interchanged with integration.

Solution. For $0 \le x < 1$, $x^n \to 0$, while $f_n(1) = 0$ for every n. Thus $f_n(x) \to 0$ pointwise on $[0, 1]$. For $n \ge 1$, maximize $g_n(x) = x^n(1 - x)$: its derivative is $x^{n-1}(n - (n + 1)x)$, so its maximum occurs at $x = n/(n + 1)$ and equals

$$\|f_n\|_\infty = \frac{n^n}{(n+1)^{n+1}} \le \frac{1}{n+1} \to 0.$$

Hence convergence is uniform. Moreover, $\int_0^1 f_n(x)dx = 1/(n + 1) - 1/(n + 2) \to 0 = \int_0^1 \lim f_n(x)dx$, so the two operations agree here.

Exercise 15. (2026). For $x \in [0, 1]$, let $f_n(x) = x^2/(n^2 + x)$. Study uniform convergence of the series $\displaystyle\sum_{n=1}^{\infty} f_n(x)$ on $[0, 1]$.

Solution. For every $x \in [0, 1]$, $0 \le x^2/(n^2 + x) \le 1/n^2$. Since $\sum_{n=1}^{\infty} 1/n^2$ converges, the Weierstrass M-test implies that $\sum_{n=1}^{\infty} x^2/(n^2 + x)$ converges uniformly on $[0, 1]$.

Exercise 16. (2024). Let $f_n(x) = (-1)^n x^{2n}$, $n \in \mathbb{N}$, $x \in \mathbb{R}$. (1) Show that $\sum f_n(x)$ converges pointwise on $(-1, 1)$. (2) Study uniform convergence on $[-t, t]$, $0 < t < 1$. (3) Using $\sum_{n=0}^{\infty} f_n(x) = 1/(1 + x^2)$, deduce $\sum_{n=0}^{\infty}(-1)^n/(2n + 1)$.

Solution. For fixed x, $\sum |f_n(x)| = \sum x^{2n}$ is geometric with ratio x^2; it converges for $|x| < 1$. At $|x| = 1$, the terms of the original series do not tend to zero. Thus the function series converges absolutely on $(-1, 1)$.
For $x \in [-t, t]$, $|f_n(x)| \le t^{2n}$ and $\sum t^{2n}$ converges since $0 < t < 1$. The Weierstrass M-test

gives normal, hence uniform, convergence on $[-t,t]$.
Integrating $\sum(-1)^n x^{2n} = 1/(1+x^2)$ from 0 to $t<1$, term-by-term integration gives $\sum(-1)^n t^{2n+1}/(2n+1) = \arctan t$. Letting $t \to 1^-$ yields $\boxed{\sum_{n=0}^{\infty}(-1)^n/(2n+1) = \pi/4}$.

Exercise 17. (2016). Let $f_n(x) = \dfrac{x}{x+n}$ for $x \geq 0$. Study pointwise convergence on $[0,1]$ and on $[1,\infty)$, and determine whether convergence is uniform on each interval.

Solution. For every fixed $x \geq 0$,

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x}{x+n} = 0,$$

so convergence is pointwise to zero on both intervals. On $[0,1]$,

$$0 \leq \sup_{x\in[0,1]} |f_n(x)| = \sup_{x\in[0,1]} \frac{x}{x+n} \leq \frac{1}{n} \to 0.$$

Thus convergence is uniform on $[0,1]$. On $[1,\infty)$, for each fixed n the function $x/(x+n)$ increases to 1 as $x \to \infty$, so

$$\sup_{x\in[1,\infty)} |f_n(x) - 0| = 1$$

for every n. This supremum does not tend to zero; convergence is not uniform on $[1,\infty)$.

Exercise 18. (2026). For $x \in [0,1]$, let $f_n(x) = \dfrac{x^2}{n^2+x}$. Prove that the series of functions $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on $[0,1]$.

Solution. For all $x \in [0,1]$ and $n \geq 1$,

$$0 \leq \frac{x^2}{n^2+x} \leq \frac{1}{n^2}.$$

The numerical series $\sum_{n=1}^{\infty} 1/n^2$ converges. The Weierstrass M-test therefore shows that $\sum_{n=1}^{\infty} x^2/(n^2+x)$ converges normally, and in particular uniformly, on $[0,1]$.

Chapter 5

Fourier Series

5.1 Trigonometric Fourier Series

Definition – Periodic Function

A function $f : \mathbb{R} \to \mathbb{R}$ is $2T$**-periodic** if $f(t+2T) = f(t)$ for all t. A $2T$-periodic function that is piecewise continuous on $[-T, T]$ is fully determined by its values on any interval of length $2T$.

Definition – Fourier Series Coefficients

For a $2T$-periodic function $f(t)$, its Fourier series is:

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi t}{T}\right) + b_n \sin\left(\frac{n\pi t}{T}\right) \right],$$

where the Euler-Fourier coefficients are:

$$a_n = \frac{1}{T}\int_{-T}^{T} f(t)\cos\left(\frac{n\pi t}{T}\right) dt, \quad b_n = \frac{1}{T}\int_{-T}^{T} f(t)\sin\left(\frac{n\pi t}{T}\right) dt.$$

Definition – Dirichlet Conditions

f satisfies the **Dirichlet conditions** on $[-T, T]$ if f is piecewise continuous and has, on any bounded interval, only finitely many extrema and discontinuities, with finite left and right limits everywhere.

Theorem – Dirichlet Convergence Theorem

If f is $2T$-periodic and satisfies the Dirichlet conditions, then its Fourier series converges at every t to

$$\frac{f(t^-) + f(t^+)}{2},$$

which equals $f(t)$ at every point of continuity.

5.2 Even and Odd Functions

Property – Fourier Series of Even and Odd Functions

If f is **even**, then $b_n = 0$ for all n and

$$a_n = \frac{2}{T}\int_0^T f(t)\cos\left(\frac{n\pi t}{T}\right)dt \qquad \text{(a } \textit{cosine series}\text{)}.$$

If f is **odd**, then $a_n = 0$ for all n and

$$b_n = \frac{2}{T}\int_0^T f(t)\sin\left(\frac{n\pi t}{T}\right)dt \qquad \text{(a } \textit{sine series}\text{)}.$$

Proposition – Linearity of Fourier Coefficients

The map $f \mapsto (a_n, b_n)$ is linear: the Fourier coefficients of $\alpha f + \beta g$ are α times those of f plus β times those of g.

Theorem – Parseval's Theorem

If f is $2T$-periodic and square-integrable on $[-T, T]$, then

$$\frac{1}{T}\int_{-T}^{T} |f(t)|^2\,dt = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right).$$

Example 1 – Fourier Series of a Square Wave

Let f be 2π-periodic, odd, with $f(t) = 1$ for $0 < t < \pi$ (a square wave). Since f is odd, $a_n = 0$ and

$$b_n = \frac{2}{\pi}\int_0^{\pi}\sin(nt)\,dt = \frac{2}{\pi}\cdot\frac{1-\cos(n\pi)}{n} = \begin{cases} \dfrac{4}{n\pi}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}.$$

Hence

$$f(t) \sim \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{\sin\big((2k+1)t\big)}{2k+1}.$$

Example 2 – Fourier Series of a Sawtooth Wave

Let $f(t) = t$ on $(-\pi, \pi)$, extended 2π-periodically. Since f is odd, $a_n = 0$ and

$$b_n = \frac{2}{\pi}\int_0^{\pi} t\sin(nt)\,dt = \frac{2}{\pi}\left[-\frac{t\cos(nt)}{n}\right]_0^{\pi} + \frac{2}{n\pi}\int_0^{\pi}\cos(nt)\,dt = \frac{2(-1)^{n+1}}{n}.$$

So $f(t) \sim 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(nt)$.

5.3 Further Fourier Concepts

Definition – Orthogonality

On $[-\pi, \pi]$, the trigonometric system is orthogonal in the sense that

$$\int_{-\pi}^{\pi} \cos(nx)\cos(mx)\,dx = 0 \quad (n \neq m),$$

$$\int_{-\pi}^{\pi} \sin(nx)\sin(mx)\,dx = 0 \quad (n \neq m),$$

and

$$\int_{-\pi}^{\pi} \sin(nx)\cos(mx)\,dx = 0.$$

This orthogonality is the fundamental mechanism for computing Fourier coefficients.

Definition – Complex Fourier Coefficients

For a 2π-periodic function, the complex Fourier series can be written

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx},$$

where

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx.$$

The real and complex forms are equivalent descriptions of the same expansion.

Theorem – Orthogonal Expansion Principle

If f is square-integrable on $[-\pi, \pi]$, its Fourier coefficients are obtained by projecting f onto the orthogonal functions $1, \cos(nx), \sin(nx)$. This projection viewpoint explains why each coefficient isolates one frequency.

Property – Parity Simplifications

For an even function, all sine coefficients vanish. For an odd function, the constant coefficient and all cosine coefficients vanish. Consequently, one may replace integrals over $[-\pi, \pi]$ by twice the corresponding integral over $[0, \pi]$.

5.4 Half-Range Fourier Series

Definition – Fourier Cosine Series

For a function defined on $[0, L]$, its cosine expansion is

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right),$$

where

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx.$$

Definition – Fourier Sine Series

The sine expansion on $[0, L]$ is

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right),$$

where

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx.$$

Example 1 – Sine Series of $f(x) = x$

On $(0, \pi)$,

$$b_n = \frac{2}{\pi}\int_0^{\pi} x\sin(nx)\,dx = \frac{2(-1)^{n+1}}{n}.$$

Hence

$$x = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\sin(nx), \qquad 0 < x < \pi.$$

5.5 Differentiation and Integration of Fourier Series

Proposition – Formal Differentiation

When the required regularity and convergence hypotheses are satisfied, a Fourier series may be differentiated term by term. If

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx),$$

then formally

$$f'(x) \sim \sum_{n=1}^{\infty}(-na_n \sin nx + nb_n \cos nx).$$

Proposition – Formal Integration

Under the usual convergence conditions,

$$\int f(x)\,dx$$

may be obtained by integrating the Fourier series term by term. Integration improves the decay of coefficients and often yields a more rapidly convergent series.

5.6 Applications

Example 1 – Solving a Simple Periodic Approximation

Suppose a periodic signal is represented by

$$f(t) = 3 + 2\cos t - \sin(2t).$$

Its average value over a period is 3, while the oscillatory components occur at frequencies 1 and 2. The Fourier representation separates the constant component from the different frequencies, which is the basis of harmonic analysis.

Theorem – Parseval Identity

If f is square-integrable and

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx),$$

then

$$\frac{1}{\pi}\int_{-\pi}^{\pi} |f(x)|^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

5.7 Further Examples and Applications

Definition – Fundamental Frequency

If f has period $2T$, its fundamental angular frequency is

$$\omega_0 = \frac{\pi}{T}.$$

The harmonics occur at integer multiples $n\omega_0$.

Property – Translation of a Periodic Function

A translation of a periodic function changes the phases of its Fourier coefficients but does not change its fundamental frequency. This is one reason Fourier analysis is useful for signals that are shifted in time.

Example 1 – Square Wave

Let

$$f(x) = \begin{cases} 1, & 0 < x < \pi, \\ -1, & -\pi < x < 0, \end{cases}$$

extended periodically. The function is odd, so only sine terms occur. One obtains

$$b_n = \frac{4}{n\pi} \quad \text{for odd } n, \qquad b_n = 0 \quad \text{for even } n.$$

Hence

$$f(x) = \frac{4}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right).$$

Example 2 – Sawtooth-Type Function

For $f(x) = x$ on $(-\pi, \pi)$, the function is odd and

$$b_n = \frac{2}{\pi}\int_{-\pi}^{\pi} x \sin(nx)\, dx = \frac{2(-1)^{n+1}}{n}.$$

Therefore

$$x = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\sin(nx), \qquad -\pi < x < \pi.$$

At the endpoints, the Fourier series converges to the midpoint of the jump.

5.8 Chapter Review

Fourier analysis decomposes a periodic function into orthogonal harmonic components. Before computing coefficients, check parity and the chosen interval. At discontinuities, the Fourier series converges to the midpoint of the left and right limits under standard Dirichlet conditions. Parseval's identity connects the energy of a function with the squares of its Fourier coefficients.

5.9 Worked Exercises

Exercise 1. Fourier Series of $|t|$ and a Sum of a Numerical Series. Let $f(t) = |t|$ on $[-\pi, \pi]$, extended 2π-periodically. Find its Fourier series and deduce the value of $\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}$.

Solution. f is even, so $b_n = 0$. Compute

$$a_0 = \frac{2}{\pi}\int_0^{\pi} t\,dt = \pi, \qquad a_n = \frac{2}{\pi}\int_0^{\pi} t\cos(nt)\,dt = \frac{2}{\pi}\cdot\frac{\cos(n\pi)-1}{n^2} = \begin{cases} -\dfrac{4}{\pi n^2}, & n \text{ odd} \\ 0, & n \text{ even.} \end{cases}$$

Hence

$$f(t) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{\cos\big((2k+1)t\big)}{(2k+1)^2}.$$

Since f is continuous, the Dirichlet theorem gives equality at every t. Setting $t = 0$: $f(0) = 0 = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}$, so

$$\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} = \frac{\pi^2}{8}.$$

Exercise 2. Using Parseval to Sum a Series. Using the square-wave example ($f(t) = 1$ on $(0,\pi)$, $f(t) = -1$ on $(-\pi, 0)$, 2π-periodic), apply Parseval's theorem to find $\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}$.

Solution. Here $\frac{1}{\pi}\int_{-\pi}^{\pi}|f(t)|^2\,dt = \frac{1}{\pi}\cdot 2\pi = 2$. From the earlier computation $b_n = \frac{4}{n\pi}$ for odd n, zero otherwise, and $a_n = 0$ for all n. Parseval gives

$$2 = \sum_{k=0}^{\infty}\left(\frac{4}{(2k+1)\pi}\right)^2 = \frac{16}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2},$$

so $\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} = \frac{2\pi^2}{16} = \frac{\pi^2}{8}$, confirming the previous exercise.

Exercise 3. Fourier Coefficients of $\cos^2(t)$. Find the Fourier series of $f(t) = \cos^2(t)$ on $[-\pi, \pi]$.

Solution. Use the identity $\cos^2(t) = \frac{1}{2} + \frac{1}{2}\cos(2t)$. This is already a finite trigonometric sum, so it is its own Fourier series: $a_0 = 1$, $a_2 = \frac{1}{2}$, all other $a_n, b_n = 0$.

5.9.1 Additional Worked Exercises

Exercise 4. Fourier Series of a Constant. Find the Fourier series of $f(x) = 3$ on $[-\pi, \pi]$.

Solution. The function is even. We have

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} 3\,dx = 6,$$

and for $n \geq 1$,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} 3\cos(nx)\,dx = 0, \qquad b_n = 0.$$

Thus $f(x) = a_0/2 = 3$, as expected.

Exercise 5. Fourier Series of x^2. Find the Fourier series of $f(x) = x^2$ on $[-\pi, \pi]$.

Solution. The function is even, so $b_n = 0$. Integration by parts twice gives

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{2\pi^2}{3}, \qquad a_n = \frac{4}{\pi}\frac{(-1)^n}{n^2}.$$

Hence

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(nx).$$

Exercise 6. Using Parseval. Use the Fourier series of $f(x) = x$ on $[-\pi, \pi]$ to find

$$\sum_{n=1}^{\infty}\frac{1}{n^2}.$$

Solution. For $f(x) = x$, $a_0 = a_n = 0$ and $b_n = 2(-1)^{n+1}/n$. Parseval gives

$$\frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \sum_{n=1}^{\infty}\frac{4}{n^2}.$$

The left side equals $2\pi^2/3$, so

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

Exercise 7. Half-Range Cosine Series. Find the cosine coefficients of $f(x) = x$ on $[0, \pi]$.

Solution. We have

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\,dx = \pi.$$

For $n \geq 1$,

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos(nx)\,dx = \frac{2}{\pi}\frac{(-1)^n - 1}{n^2}.$$

Thus $a_n = 0$ for even n and $a_n = -4/(\pi n^2)$ for odd n.

Exercise 8. Square-Wave Coefficient. Compute the coefficient b_n for the square wave above.

Solution. Since f is odd,

$$b_n = \frac{2}{\pi}\int_0^{\pi}\sin(nx)\,dx = \frac{2}{\pi}\frac{1 - \cos(n\pi)}{n}.$$

Thus $b_n = 0$ for even n and $b_n = 4/(n\pi)$ for odd n.

Exercise 9. A Fourier Value. Use the Fourier series of x at $x = \pi/2$ to obtain an alternating series.

Solution. Substituting $x = \pi/2$ gives

$$\frac{\pi}{2} = 2\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right).$$

Hence
$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}.$$

Exercise 10. Review Exercise. Find the Fourier coefficients of $f(x) = 1 + \cos x$ on $[-\pi, \pi]$.

Solution. Here $a_0 = 2$, $a_1 = 1$, and all other a_n, b_n vanish. Thus the Fourier series is simply $1 + \cos x$.

Exercise 11. Parity. Explain why the Fourier sine coefficients of an even function vanish.

Solution. If f is even, then $f(x)\sin(nx)$ is odd. The integral of an odd function over $[-\pi, \pi]$ is zero. Hence every $b_n = 0$.

5.9.2 Exercises from Previous Mathematics 3 Examinations

Exercise 12. (2023). Let f be the 2π-periodic function whose values on $[0, 2\pi]$ are $f(x) = (x - \pi)^2$. Find its Fourier series and deduce the sums $\sum_{n=1}^{\infty} 1/n^2$ and $\sum_{n=1}^{\infty} (-1)^n/n^2$.

Solution. Set $t = x - \pi$. On $[-\pi, \pi]$, the function becomes t^2, whose Fourier expansion is
$$t^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n \cos(nt)}{n^2}.$$
Since $\cos(n(x - \pi)) = (-1)^n \cos(nx)$, the two factors $(-1)^n$ cancel. Thus, on $0 < x < 2\pi$,
$$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2}.$$
At $x = 0$, the periodic function has value π^2 from both endpoint values, and the series gives
$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{1}{n^2}, \qquad \text{so} \qquad \boxed{\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}}.$$
At $x = \pi$, $f(\pi) = 0$ and $\cos(n\pi) = (-1)^n$, hence
$$0 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}, \qquad \text{so} \qquad \boxed{\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}}.$$

The following problems have been transcribed from the supplied collection of Mathematics 3 examination papers. Repeated versions of essentially the same problem have been combined; no solutions are included in this section.

Exercise 13. (2013). Develop the Fourier series of the 2π-periodic function
$$f(x) = \begin{cases} \dfrac{x}{2}, & x \in (0, \pi), \\ \pi + \dfrac{x}{2}, & x \in (-\pi, 0). \end{cases}$$
Use the coefficient definitions
$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx, \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx, \quad n \geq 1.$$

Solution. The mean coefficient is $a_0 = \frac{1}{2\pi}\left(\int_{-\pi}^{0}(\pi + x/2)dx + \int_0^{\pi} x/2\,dx\right) = \pi/2$. For $n \geq 1$, direct integration gives

$$a_n = 0, \qquad b_n = -\frac{1}{n}.$$

Consequently, using the convention $f(x) = a_0 + \sum_{n\geq 1}(a_n \cos nx + b_n \sin nx)$, its Fourier series is

$$f(x) \sim \frac{\pi}{2} - \sum_{n=1}^{\infty} \frac{\sin(nx)}{n}.$$

At jump points the Fourier series converges to the average of the left and right limits.

Exercise 14. (2023). Let f be the even, 2π-periodic function defined by

$$f(x) = |x|, \qquad x \in [-\pi, \pi].$$

1. Sketch the graph of f on $[-2\pi, 2\pi]$.
2. Calculate its Fourier coefficients a_0, a_n and b_n.
3. Write the associated Fourier series and deduce
$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}.$$

Solution. The function is even, so $b_n = 0$. Also $a_0 = \frac{1}{\pi}\int_0^{\pi} x\,dx = \pi/2$. For $n \geq 1$,

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos(nx)dx = \frac{2}{\pi}\frac{(-1)^n - 1}{n^2}.$$

Thus $a_n = 0$ for even n and $a_n = -4/(\pi n^2)$ for odd n, giving

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{\cos((2n+1)x)}{(2n+1)^2}.$$

At $x = 0$, $0 = \pi/2 - (4/\pi)\sum(2n+1)^{-2}$, so $\boxed{\sum_{n=0}^{\infty}(2n+1)^{-2} = \pi^2/8}$.

Exercise 15. (2024). Let f be an even, 2π-periodic function defined on $[0, \pi]$ by

$$f(x) = \pi - x.$$

1. Sketch the graph of f on $[-3\pi, 3\pi]$.
2. Find the Fourier series associated with f.
3. Deduce the values of
$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \qquad \text{and} \qquad \sum_{n=0}^{\infty} \frac{1}{(2n+1)^4}.$$

Solution. The even extension is $f(x) = \pi - |x|$ on $[-\pi, \pi]$. Hence $b_n = 0$, $a_0 = \frac{1}{\pi}\int_0^\pi (\pi - x)dx = \pi/2$, and

$$a_n = \frac{2}{\pi}\int_0^\pi (\pi - x)\cos(nx)dx = \frac{2(1-(-1)^n)}{\pi n^2}.$$

Therefore $f(x) = \pi/2 + (4/\pi)\sum_{n=0}^\infty \cos((2n+1)x)/(2n+1)^2$. At $x = 0$, $\pi = \pi/2 + (4/\pi)S_2$, so $S_2 = \pi^2/8$. Parseval gives $\frac{1}{\pi}\int_{-\pi}^\pi f^2 = \frac{a_0^2}{2} + \sum a_n^2$. Since $\int_0^\pi (\pi - x)^2 dx = \pi^3/3$, this yields $\pi^2/3 = \pi^2/8 + (16/\pi^2)S_4$, hence $\boxed{S_4 = \pi^4/96}$.

Exercise 16. (2025). Let $f(x) = x$ for $x \in [-\pi, \pi]$, extended periodically with period 2π.

1. Find the Fourier series of f and use it to calculate
$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}.$$
2. Using Parseval's identity, calculate
$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$
3. Deduce the value of
$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}.$$

Solution. Since f is odd, $a_0 = a_n = 0$, and

$$b_n = \frac{1}{\pi}\int_{-\pi}^\pi x\sin(nx)dx = \frac{2(-1)^{n+1}}{n}.$$

Thus $x = 2\sum_{n=1}^\infty (-1)^{n+1}\sin(nx)/n$ for $-\pi < x < \pi$. Taking $x = \pi/2$ gives $\sum_{n=0}^\infty (-1)^n/(2n+1) = \pi/4$. Parseval gives $\frac{1}{\pi}\int_{-\pi}^\pi x^2 dx = \sum b_n^2$, so $2\pi^2/3 = 4\sum 1/n^2$, and $\sum 1/n^2 = \pi^2/6$. Removing the even terms gives $\sum_{n=0}^\infty 1/(2n+1)^2 = (1 - 1/4)\pi^2/6 = \pi^2/8$.

Exercise 17. (2026). Let $f(x) = x^2$ for $x \in [-\pi, \pi]$, extended periodically with period 2π.

1. Write the Fourier series associated with f.
2. Calculate
$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2}.$$
3. Deduce the value of
$$\sum_{n=1}^{\infty} \frac{1}{n^4}.$$

You may use Parseval's identity:

$$\frac{1}{\pi}\int_{-\pi}^{\pi} |f(x)|^2\, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

Solution. The function is even, so $b_n = 0$, and $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx = 2\pi^2/3$. Two integrations by parts give $a_n = 4(-1)^n/n^2$. Therefore

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n \cos(nx)}{n^2}.$$

At $x = 0$, $0 = \pi^2/3 + 4\sum_{n=1}^{\infty}(-1)^n/n^2$, so the alternating sum from $n = 0$ is $1 - \pi^2/12$. Parseval gives $\frac{1}{\pi}\int_{-\pi}^{\pi} x^4 dx = \frac{a_0^2}{2} + \sum a_n^2$, hence $2\pi^4/5 = 2\pi^4/9 + 16\sum 1/n^4$. Thus $\boxed{\sum_{n=1}^{\infty} 1/n^4 = \pi^4/90}$.

Exercise 18. (2026). Let $f(x) = |x|$ on $[-\pi, \pi]$.

1. Compute the Fourier series of f.
2. Use Parseval's identity to find
$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4}.$$

Solution. As above, $|x| = \pi/2 - (4/\pi)\sum_{n=0}^{\infty} \cos((2n+1)x)/(2n+1)^2$. Parseval gives

$$\frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{a_0^2}{2} + \sum a_n^2.$$

The left side is $2\pi^2/3$, while $a_0 = \pi/2$ and the nonzero coefficients are $a_{2n+1} = -4/(\pi(2n+1)^2)$. Hence $2\pi^2/3 = \pi^2/8 + (16/\pi^2)\sum_{n=0}^{\infty}(2n+1)^{-4}$, so the sum is $\boxed{\pi^4/96}$.

Exercise 19. (2023). Let f be the 2π-periodic function defined on $[0, 2\pi)$ by

$$f(x) = (x - \pi)^2.$$

1. Sketch the graph of f on $[-2\pi, 2\pi)$.
2. Calculate the Fourier coefficients of f.
3. Deduce the values of
$$\sum_{n=1}^{\infty} \frac{1}{n^2} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}.$$

Solution. Put $t = x - \pi$ on $[0, 2\pi)$. The function is the translate of t^2 on $[-\pi, \pi]$, so its Fourier series is

$$(x-\pi)^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{\cos(n(x-\pi))}{n^2} = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n \cos(nx)}{n^2}.$$

At $x = \pi$ the value is zero, giving $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$. At $x = 0$, $\pi^2 = \pi^2/3 + 4\sum(-1)^n/n^2$, hence $\sum(-1)^n/n^2 = \pi^2/6$.

Exercise 20. (2026). Let f be the 2π-periodic function defined by

$$f(x) = \pi - x, \qquad x \in [-\pi, \pi].$$

1. Sketch the graph of f on $[-3\pi, 3\pi]$.
2. Write the Fourier series associated with f.
3. Deduce the sum
$$\sum_{n=1}^{\infty} \frac{1}{(2n+1)^2}.$$

Solution. The function $f(x) = \pi - x$ is the standard sawtooth. Its coefficients are $a_0 = \pi$, $a_n = 0$, and $b_n = 2/n$ for $n \geq 1$. Thus

$$\pi - x = \pi + 2\sum_{n=1}^{\infty} \frac{\sin(nx)}{n}, \qquad -\pi < x < \pi.$$

The requested odd-square sum is obtained from the standard identity $\sum_{n=0}^{\infty}(2n+1)^{-2} = \pi^2/8$ (or by the even/odd decomposition of $\sum n^{-2} = \pi^2/6$). Note: the exam's displayed lower index should be checked against the source scan because it appears to start at $n = 1$ while the standard odd-term formula starts at $n = 0$.

Exercise 21. (2024). Let f be even and 2π-periodic, with $f(x) = \pi - x$ for $x \in [0, \pi]$. Sketch the graph on $[-3\pi, 3\pi]$, find its Fourier series, and deduce $\sum_{n=0}^{\infty} 1/(2n+1)^2$.

Solution. By evenness, $f(x) = \pi - |x|$ on $[-\pi, \pi]$, and $b_n = 0$. With the convention $f(x) \sim a_0/2 + \sum_{n\geq 1}(a_n \cos nx + b_n \sin nx)$,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \pi.$$

For $n \geq 1$, integration by parts gives

$$a_n = \frac{2}{\pi}\int_0^{\pi} (\pi - x)\cos(nx)\,dx = \frac{2(1-(-1)^n)}{\pi n^2}.$$

Thus $a_n = 0$ for even n and $a_n = 4/(\pi(2k+1)^2)$ for $n = 2k+1$. Therefore

$$f(x) = \frac{\pi}{2} + \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{\cos((2k+1)x)}{(2k+1)^2}.$$

At $x = 0$, $f(0) = \pi$, so $\pi = \pi/2 + (4/\pi)\sum_{k=0}^{\infty}(2k+1)^{-2}$, and $\boxed{\sum_{k=0}^{\infty}(2k+1)^{-2} = \pi^2/8}$.

Exercise 22. (2023). Let $f(x) = |x|$ on $[-\pi, \pi]$, extended periodically with period 2π. Find its Fourier series and deduce

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}.$$

Solution. The function is even, so $b_n = 0$. With the convention $f(x) = a_0 + \sum_{n\geq 1}(a_n \cos nx + b_n \sin nx)$,

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |x|\,dx = \frac{\pi}{2},$$

and

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\cos(nx)\,dx = \frac{2}{\pi}\int_0^{\pi} x\cos(nx)\,dx = \frac{2}{\pi}\frac{(-1)^n - 1}{n^2}.$$

Thus $a_n = 0$ for even n and $a_n = -4/(\pi n^2)$ for odd n. The Fourier series is

$$|x| \sim \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\cos((2n+1)x)}{(2n+1)^2}.$$

At $x = 0$ the function is continuous and equals zero, so

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}, \qquad \boxed{\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{\pi^2}{8}}.$$

Exercise 23. (2018). For the 2π-periodic function $f(x) = (x-\pi)^2$ on $[0, 2\pi)$, find a Fourier series and deduce the value of $\sum_{n=1}^{\infty} 1/n^2$.

Solution. Put $t = x - \pi$, so $t \in [-\pi, \pi)$. The standard Fourier series of t^2 on $[-\pi, \pi]$ is

$$t^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n\cos(nt)}{n^2}.$$

Since $\cos(n(x-\pi)) = (-1)^n\cos(nx)$, the two factors $(-1)^n$ cancel. Therefore

$$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{\cos(nx)}{n^2}.$$

At $x = 0$, the periodic function is continuous and $f(0) = \pi^2$. Substitution gives

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2},$$

so $\boxed{\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6}$.

Chapter 6

Fourier and Laplace Transforms

6.1 The Fourier Transform

Definition – Fourier Transform & Inverse

The **Fourier Transform** of $f(t) \in L^1(\mathbb{R})$ is defined by:

$$\mathcal{F}\{f(t)\}(\omega) = \hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt.$$

The **inverse Fourier transform** recovers f from $\hat{f}$:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{i\omega t}\, d\omega.$$

Property – Properties of the Fourier Transform

For suitable f, g and constants $\alpha, \beta, a \neq 0, t_0$:

- **Linearity:** $\mathcal{F}\{\alpha f + \beta g\} = \alpha \hat{f} + \beta \hat{g}$.
- **Time shift:** $\mathcal{F}\{f(t - t_0)\}(\omega) = e^{-i\omega t_0} \hat{f}(\omega)$.
- **Modulation:** $\mathcal{F}\{e^{i\omega_0 t} f(t)\}(\omega) = \hat{f}(\omega - \omega_0)$.
- **Scaling:** $\mathcal{F}\{f(at)\}(\omega) = \frac{1}{|a|} \hat{f}\left(\frac{\omega}{a}\right)$.
- **Derivative:** $\mathcal{F}\{f'(t)\}(\omega) = i\omega \hat{f}(\omega)$.
- **Convolution:** $\mathcal{F}\{(f * g)(t)\}(\omega) = \hat{f}(\omega)\, \hat{g}(\omega)$, where $(f*g)(t) = \int_{-\infty}^{\infty} f(\tau) g(t-\tau)\, d\tau$.

Example 1 – Fourier Transform of a Rectangular Pulse

Let $f(t) = 1$ for $|t| \leq a$ and 0 otherwise. Then

$$\hat{f}(\omega) = \int_{-a}^{a} e^{-i\omega t}\, dt = \frac{e^{-i\omega a} - e^{i\omega a}}{-i\omega} = \frac{2\sin(a\omega)}{\omega}.$$

Example 2 – Fourier Transform of a Decaying Exponential

Let $f(t) = e^{-a|t|}$, $a > 0$. Then

$$\hat{f}(\omega) = \int_{-\infty}^{0} e^{at} e^{-i\omega t}\, dt + \int_{0}^{\infty} e^{-at} e^{-i\omega t}\, dt = \frac{1}{a - i\omega} + \frac{1}{a + i\omega} = \frac{2a}{a^2 + \omega^2}.$$

6.2 The Laplace Transform

Definition – Laplace Transform

The Laplace Transform of $f(t)$ defined for $t \geq 0$ is:

$$\mathcal{L}\{f(t)\}(s) = F(s) = \int_{0}^{\infty} f(t) e^{-st}\, dt, \quad s \in \mathbb{C}.$$

It exists (converges) for $\mathrm{Re}(s)$ larger than some abscissa of convergence, provided f is piecewise continuous and of exponential order.

Theorem – Properties of Laplace Transform

- **Linearity:** $\mathcal{L}\{\alpha f + \beta g\} = \alpha \mathcal{L}\{f\} + \beta \mathcal{L}\{g\}$.
- **Derivative Rule:** $\mathcal{L}\{f'(t)\}(s) = sF(s) - f(0)$.
- **Second Derivative:** $\mathcal{L}\{f''(t)\}(s) = s^2 F(s) - sf(0) - f'(0)$.

Property – Further Properties of the Laplace Transform

- **First shifting theorem:** $\mathcal{L}\{e^{at} f(t)\}(s) = F(s - a)$.
- **Second shifting theorem:** For $a \geq 0$, $\mathcal{L}\{f(t - a)u(t - a)\}(s) = e^{-as} F(s)$, where u is the Heaviside step function.
- **Scaling:** $\mathcal{L}\{f(at)\}(s) = \frac{1}{a} F\left(\frac{s}{a}\right)$ for $a > 0$.
- **Multiplication by t^n:** $\mathcal{L}\{t^n f(t)\}(s) = (-1)^n F^{(n)}(s)$.
- **Convolution theorem:** $\mathcal{L}\{(f * g)(t)\}(s) = F(s)G(s)$.
- **Initial value theorem:** $f(0^+) = \lim_{s \to \infty} sF(s)$.
- **Final value theorem:** If $\lim_{t \to \infty} f(t)$ exists, then $\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$.

Proposition – Table of Elementary Laplace Transforms

$$\mathcal{L}\{1\} = \frac{1}{s}, \qquad \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \qquad \mathcal{L}\{e^{at}\} = \frac{1}{s-a},$$
$$\mathcal{L}\{\sin(\omega t)\} = \frac{\omega}{s^2+\omega^2}, \qquad \mathcal{L}\{\cos(\omega t)\} = \frac{s}{s^2+\omega^2}.$$

Example 1 – Direct Computation from the Definition

Compute $\mathcal{L}\{e^{at}\}(s)$ directly. For $\mathrm{Re}(s) > a$,

$$\mathcal{L}\{e^{at}\}(s) = \int_0^\infty e^{at}e^{-st}\,dt = \int_0^\infty e^{-(s-a)t}\,dt = \left[\frac{-e^{-(s-a)t}}{s-a}\right]_0^\infty = \frac{1}{s-a}.$$

Example 2 – Using the Shifting Theorem

Compute $\mathcal{L}\{e^{2t}\cos(3t)\}$. Since $\mathcal{L}\{\cos(3t)\} = \frac{s}{s^2+9}$, the first shifting theorem gives

$$\mathcal{L}\{e^{2t}\cos(3t)\}(s) = \frac{s-2}{(s-2)^2+9}.$$

6.3 Solving Differential Equations with the Laplace Transform

Proposition – Method

To solve a linear ODE with initial conditions: (1) apply $\mathcal{L}$ to both sides, using the derivative rules to convert derivatives into algebraic expressions in s; (2) solve the resulting algebraic equation for $F(s) = \mathcal{L}\{f\}$; (3) take the inverse transform $\mathcal{L}^{-1}$, typically via partial fractions and the table of transforms, to recover $f(t)$.

Example 1 – Solving a Second-Order ODE

Solve $y'' + 4y = 0$, $y(0) = 1$, $y'(0) = 0$. Applying $\mathcal{L}$:

$$s^2Y(s) - sy(0) - y'(0) + 4Y(s) = 0 \implies (s^2+4)Y(s) = s \implies Y(s) = \frac{s}{s^2+4}.$$

Since $\mathcal{L}\{\cos(2t)\} = \frac{s}{s^2+4}$, we conclude $y(t) = \cos(2t)$.

6.4 Additional Laplace Transform Theory

Definition – Laplace Transform

For a suitable function $f : [0, \infty) \to \mathbb{R}$, its Laplace transform is

$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)\, dt,$$

for those values of s for which the improper integral converges.

Definition – Piecewise Continuity and Exponential Order

A function is of **exponential order** if there are constants $M, c, T > 0$ such that

$$|f(t)| \leq Me^{ct} \qquad (t \geq T).$$

Piecewise continuity together with exponential order is a standard sufficient condition for the existence of the Laplace transform for sufficiently large s.

Property – Linearity

For constants α, β,

$$\mathcal{L}\{\alpha f + \beta g\} = \alpha \mathcal{L}\{f\} + \beta \mathcal{L}\{g\}.$$

Theorem – First Derivative Formula

If f satisfies the usual hypotheses,

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0), \qquad F(s) = \mathcal{L}\{f\}.$$

More generally,

$$\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - \cdots - f^{(n-1)}(0).$$

Theorem – First Shifting Theorem

If $\mathcal{L}\{f(t)\} = F(s)$, then

$$\mathcal{L}\{e^{at} f(t)\} = F(s - a).$$

Theorem – Second Shifting Theorem

If $u(t - a)$ is the Heaviside step function and

$$\mathcal{L}\{f(t)\} = F(s),$$

then

$$\mathcal{L}\{u(t - a) f(t - a)\} = e^{-as} F(s).$$

This theorem is useful for piecewise forcing terms and delayed signals.

6.5 Inverse Laplace Transforms

Proposition – Partial Fractions

Rational functions of s can often be inverted after decomposition into elementary fractions such as

$$\frac{A}{s-a}, \qquad \frac{Bs+C}{s^2+\omega^2}.$$

The decomposition should be performed before applying a transform table.

Example 1 – Inverse Transform by Partial Fractions

Find

$$\mathcal{L}^{-1}\left\{\frac{5s+7}{(s+1)(s+2)}\right\}.$$

Write

$$\frac{5s+7}{(s+1)(s+2)} = \frac{2}{s+1} + \frac{3}{s+2}.$$

Therefore

$$f(t) = 2e^{-t} + 3e^{-2t}.$$

6.6 Convolution

Definition – Convolution

For suitable functions f and g on $[0,\infty)$, their convolution is

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau.$$

Theorem – Convolution Theorem

If $F = \mathcal{L}\{f\}$ and $G = \mathcal{L}\{g\}$, then

$$\mathcal{L}\{f * g\} = F(s)G(s).$$

Consequently,

$$\mathcal{L}^{-1}\{F(s)G(s)\} = f * g.$$

6.7 Fourier Transform

Definition – Fourier Transform

For a sufficiently integrable function f on $\mathbb{R}$, one common convention is

$$\widehat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt.$$

The inverse transform is

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(\omega)e^{i\omega t}\,d\omega,$$

under suitable hypotheses.

Property – Fourier Transform Rules

Under appropriate assumptions,

$$\widehat{f'}(\omega) = i\omega\widehat{f}(\omega), \qquad \widehat{tf(t)}(\omega) = i\frac{d}{d\omega}\widehat{f}(\omega),$$

and

$$\widehat{f * g}(\omega) = \widehat{f}(\omega)\widehat{g}(\omega).$$

These rules turn differentiation and convolution into algebraic operations.

Example 1 – Transform of an Exponential Kernel

For $f(t) = e^{-at}$ on $t \geq 0$, $a > 0$,

$$\mathcal{L}\{e^{-at}\}(s) = \int_0^{\infty} e^{-(s+a)t}\,dt = \frac{1}{s+a}, \qquad s > -a.$$

6.8 Further Transform Properties

Property – Scaling

For $a > 0$,

$$\mathcal{L}\{f(at)\}(s) = \frac{1}{a}F\left(\frac{s}{a}\right), \qquad F(s) = \mathcal{L}\{f(t)\}.$$

Scaling changes the time variable and correspondingly rescales the transform variable.

Property – Multiplication by t

If $F(s) = \mathcal{L}\{f(t)\}$, then

$$\mathcal{L}\{tf(t)\} = -F'(s).$$

More generally,

$$\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s).$$

Example 1 – Transform of te^{-at}

Since

$$\mathcal{L}\{e^{-at}\} = \frac{1}{s+a},$$

we obtain

$$\mathcal{L}\{te^{-at}\} = -\frac{d}{ds}\left(\frac{1}{s+a}\right) = \frac{1}{(s+a)^2}.$$

Theorem – Initial Value Theorem

Under suitable hypotheses,

$$\lim_{t\to 0^+} f(t) = \lim_{s\to\infty} sF(s).$$

This provides a quick way of checking an inverse transform or an initial condition.

Theorem – Final Value Theorem

When the required stability hypotheses hold,

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s).$$

The theorem should not be applied blindly when poles of $sF(s)$ lie in the closed right half-plane.

Example 2 – Initial Value Check

Let

$$F(s) = \frac{1}{s(s+1)}.$$

Then

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty} \frac{1}{s+1} = 0.$$

Indeed,

$$\mathcal{L}^{-1}\{F(s)\} = 1 - e^{-t},$$

whose initial value is 0.

6.9 Chapter Review

The Laplace transform converts differentiation of functions into algebraic operations in the variable s, making it especially effective for initial-value problems. The most frequently used tools are linearity, derivative formulas, shifting, partial fractions, convolution, and differentiation with respect to s. The Fourier transform instead emphasizes frequency content and provides powerful rules for differentiation, translation, scaling, and convolution.