Chapter 1

Mathematical Reasoning

1.1 Mathematical Logic

1.1.1 Statements

A statement is a sentence which is either true or false, but not both simultaneously.

Example

a) 2 + 2 = 4 is a true statement.
b) 3 × 2 = 7 is a false statement.
c) For all x ∈ **)belong to(**R we have $x^2 \geq 0$ is a true statement.
e) For all x ∈ R we have |x| = 1 is a false assertion.

1.1.2 Logical Operations

If P is an assertion and Q is another assertion, we will define new assertions constructed from P and Q

a) **The logical operator "and" (∧) (Conjunction)**

Consider two statements P and Q.

The statement P and Q is true provided P is true and Q is true. Otherwise, P and Q is false. We summarize this in a truth table:

A	B	$A \wedge B$
True	True	True
True	False	False
False	True	False

Exemple

$a)\ (3+5=8) \wedge (3 \times 6 = 18)$ *est une assertion vraie.* ST

$b)\ (2+2=4) \wedge (2 \times 3 = 7)$ *est une assertion fausse.* SF

b) **The logical operator "or" (∨) (Disjunction)**

The statement P or Q is true provided P is true, Q is true, or both are true. Otherwise, P or Q is false. We summarize this in a truth table:

Logical Disjunction

p	q	$p \vee q$
F	F	F
F	T	T
T	F	T
T	T	T

Exemple

$a)$ $(2+2=4) \vee (3 \times 2 = 6)$ *est une assertion vraie.* **ST**

$b)$ $(2=4) \vee (4 \times 3 = 7)$ *est une assertion fausse.* **SF**

c) **Logical negation "not" "~ p"**

Logical negation is an operation on one logical value, typically the value of a proposition, that produces a value of *true* when its operand is false and a value of *false* when its operand is true. The truth table of **"not P"** also written **"~ p"**appears below:

p	$\neg p$
F	T
T	F

Exemple The negation of the assertion $3 \geq 0$ is the assertion $3 < 0$.

d) **Implication (⇒) '' If, then''**

The implication or conditional is the statement "If P then Q" and is de-

noted by $P \rightarrow Q$. The statement $P \rightarrow Q$ is often read as "P implies Q," that $P \rightarrow Q$ is false only when P is true and Q is false

p	q	$\text{Cond}(p, q)$
F	F	T
F	T	T
T	F	F
T	T	T

All of the following have the same meaning:

If P , then Q.

P implies Q.

P ⇒ Q (read P implies Q)

Q, if P .

P only if Q.

Q when (or whenever) P .

Q is necessary for P .

P is sufficient for Q.

e) **Equivalent (⇔) " if and only if"**

The statement P if and only if Q, written P ⇔ Q, is equivalent to the statement (P ⇒ Q) and (Q ⇒ P).

P ⇔ Q is true provided P and Q have the same truth value. If P and Q do not have the same truth value, then P ⇔ Q is false.

P	Q	$P \Leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

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1.1.3 Quantifiers

- For an open setence P (x), we have the propositions (∃x)P (x) which is true when there exists at least one x for which P (x) is true.

The symbol ∃ is called the existential quantifier.

∃x ∈)**belong to(** E, P (x) est une assertion vraie lorsque l'on peut trouver au moins un élément x de E pour lequel P (x) est vraie. On lit il existe x appartenant à E tel que P (x) (soit vraie).

Remark . The existential statement (∃x)P (x)

may be read as:

- "There exists x such that P (x)."
- "There exists x for which P (x)."
- "For some x, P (x).

- **(∀x)P (x) which is true when P (x) is true for every x. The symbol ∀ is called the universal quantifier.**

∀x ∈)**belong to(** E, P (x) est une assertion vraie lorsque les assertions P (x) sont vraies pour tous les éléments x de l'ensemble E. On lit : pour tout x appartenant à E, P (x) est vraie.

Remark . the universal statement (∀x)P (x) may be read as:

- "For all x, P (x)."
- "For every x, P (x)."
- "For each x, P (x)."

The symbol ∀ was chosen as an inverted A for "all."

- **Negations of quantified statements**

- The negation of for all n, P (n) is:

  there exists n such that not(P (n)).

$$(\forall x \in E, \quad P(x)) \quad \text{est} \quad \left(\exists x \in E, \quad \overline{P(x)}\right).$$

Exemple : the negation of $\left(\forall x \in \mathbb{R} : \underbrace{x^2 \geq 0}_{P(x)}\right)$ is $\exists x \in \mathbb{R} : \underbrace{x^2 < 0}_{\overline{P(x)}}$.

- The negation of there exists n such that P (n) is:

  for all n, not(P (n)).

$$(\exists x \in E, \quad P(x)) \quad \text{est} \quad \left(\forall x \in E, \quad \overline{P(x)}\right).$$

If n is divisible by 4, then n is even,

Its negation is therefore:

There is an integer n such that n is divisible by 4 and n is not even.

Exemple : the negation of $\left(\exists x \in \mathbb{R} : \underbrace{x < 0}_{P(x)}\right)$ is $\forall x \in \mathbb{R} : \underbrace{x \geq 0}_{\overline{P(x)}}$

1.1.4 Summary of Negations

The negations of common logical expressions are summarized in the following table :

statement	negation
P *and* Q	(*not* P) *or* (*not* Q)
P *or* Q	(*not* P) *and* (*not* Q)
$P \Rightarrow Q$	P *and* (*not* Q)
$P \Leftrightarrow Q$	(P *and not* Q) *or* (Q *and not* P)
$\forall n, P(n)$	$\exists n$ such that *not*$(P(n))$
$\exists n$ such that $P(n)$	$\forall n$, *not*$(P(n))$

1.2Proof Methods

1.2.1 Direct Proofs

We want to show that the assertion P ⇒ implies Q is true. We assume that P is true and we then show that Q is true.

Example: Let a, b ∈ R. Show that $a = b \Rightarrow \frac{a+b}{2} = b.$

Let's take a = b, then $\frac{a}{2} = \frac{b}{2}$, SO $\frac{a}{2} + \frac{b}{2} = \frac{b}{2} + \frac{b}{2}.$ as well as $\frac{a+b}{2} = b.$

1.2.2 Proof by contrapositive

Proof by contrapositive is based on the following equivalence :

$$(P \Rightarrow Q) \Leftrightarrow (\sim Q \Rightarrow \sim P)$$

Recall that the statement P ⇒ Q has the same truth value as its contrapositive (not Q) ⇒ (not P).
Therefore, if you wish to prove that P ⇒ Q is true, you may prove instead that its contrapositive is true.
This is called proof by contrapositive: you suppose not Q as your hypothesis and show that, under that assumption, not P is true

Exemple :
Let x ∈(belonged) R. Show that

$$\underbrace{(x \neq 2 \;\; et \;\; x \neq -2)}_{P} \Rightarrow \underbrace{(x^2 \neq 4)}_{Q}.$$

By contraposition this is equivalent :

$$\underbrace{(x^2 = 4)}_{\overline{Q}} \Rightarrow \underbrace{(x = 2 \;\; ou \;\; x = -2)}_{\overline{P}}.$$

indeed, let's take $x^2 = 4$, then (x − 2) (x + 2) = 0, therefore x = 2 or x = −2.

1.2.3 **Proof by Contradiction(**absurdity)

to show P ⇒ Q, is based on the following principle:

We assume both that P is true and that Q is false and we look for a contradiction. So if P is true then Q must be true and therefore P ⇒ Q is true.

Exemple :

Let a, b > 0. Show that if $\frac{a}{1+b} = \frac{b}{1+a} \Rightarrow a = b.$

We reason absurdly by supposing that

$$\frac{a}{1+b} = \frac{b}{1+a} \quad et \quad a \neq b.$$

We have

$$\begin{aligned}\left(\frac{a}{1+b} = \frac{b}{1+a}\right) &\Leftrightarrow a(a+1) = b(b+1)\\ &\Leftrightarrow a^2 - b^2 = -(a-b)\\ &\Leftrightarrow (a-b)(a+b) = -(a-b)\end{aligned}$$

This is equivalent

$$(a-b)(a+b) = -(a-b) \quad et \quad a-b \neq 0.$$

so by dividing by (a – b) we obtain

$$a + b = -1.$$

The sum of two positive numbers cannot be negative. We get a contradiction.

1.2.4 Proof by Induction **(recurrence)**

The principle of Proof by Induction allows us to show that an assertion P (n), depending on n, is true for all n ∈ N. The proof by induction takes place in two steps:

i) We prove P (0) is true.

On prouve P (0) est vraie.

ii) We assume $n \geq 0$ given with P (n) true, and we then demonstrate that the assertion P (n + 1) is true.

Finally in the conclusion, we recall that by the principle of recurrence P (n) is true for all $n \in N$.

Exemple :

Prove $2^n > n+4$ for $n \geq 3, n \in \mathbb{N}$.

Solution

Let $n=3$. Then $2^3 > 3+4$ is true since clearly $8>7$. Thus the statement is true for $n=3$.

Assume that $2^n > n+4$ is true for some $n=k$.

We will show that $2^{k+1} > (k+1)+4$.

Consider $2^{k+1} = 2 \cdot 2^k > 2 \cdot (k+4) = 2k+8$.

Since $2k > k+1$ and $8>4$, we have $2k+8 > (k+1)+4$.

Thus the statement is true for all $n=k$.

By induction, $2^n > n+4$ for all $n \geq 3, n \in \mathbb{Z}$.□

Exemple 2 Show that for all $n \in N$: $2^n > n$.

Let us note P (n): $2^n > n$, for all $n \in N$.

We will demonstrate by induction that P (n) is true for all $n \in N$.

i) For n = 0 we have $2^0 = 1 > 0$, so P (0) is true.

ii) Let $n \in N$, suppose P(n) is true. We will show that P (n + 1) is true.

$2^{n+1} = 2^n + 2^n$

$> n + 2^n$, because by P (n) we know that $2^n > n$,

$\geq n + 1$, because $2^n \geq 1$

So P(n + 1) is true.